# Complexity of the Sequence of some Families of Graphs Based on Tridiminished icosahedron Graph

**Ahmad Asiri[1] and Salama Nagy Daoud[2,3]**

[1]Department of Mathematics, Applied College at Mahail Aseer, King Khalid University, Saudia Arabia

[2]Department of Mathematics, Faculty of Science, Taibah University, Al-Madinah Al-Nunawara 30001, Saudi Arabia.

[3]Department of Mathematics and Computer Sciences, Faculty of Science, Menoufia University, Shebin El Kom 32511, Egypt.

amhasiri@kku.edu.sa & salamadaoud@gmail.com

## Abstract

Tridiminished icosahedron graph is one of nonahedral graphs. In this work, using knowledge of difference equations we drive the explicit formulas for the number of spanning trees in the sequence of some new families of graphs of average degree four based on Tridiminished icosahedron graph by electrically equivalent transformations and rules of weighted generating function. Finally, we compare the entropy of our graphs with other studied graphs with average degree being 4.

**Keywords**: Number of spanning trees; Tridiminished icosahedron graph; Electrically equivalent transformations.

**Mathematics Subject Classification**: 05C30, 05C50, 05C63.

## 1. Introduction

Deriving closed formulae of the number of spanning trees for various graphs has attracted the attention of a lot of researchers. The importance of this research line is in fact due to:

1- Enumerating specific chemical isomers,
2- Counting the number of Eulerian circuits in a graph,
3- Solving some computationally hard problems such as the Steiner tree problem and traveling salesman problem.
4- Deriving formulas for different type of graphs can be helpful in identifying those graphs that contain the maximum number of spanning trees. Such an investigation has practical consequences related to network reliability, [1-12].

A spanning subgraph of a graph $G(V,E)$ is a subgraph with vertex set $V$. A spanning tree is a spanning subgraph. $\tau(G)$ denote the complexity (Number of spanning trees of a graph $G$).

There exist various methods for finding this number. Kirchhoff [13] gave the famous matrix tree theorem: if $D$ is the diagonal matrix of the degrees of $G$ and $A$ denote the adjacency matrix of $G$, Kirchhoff matrix $L = D - A$ has all of its cofactors equal to $\tau(G)$.

Another method to count the complexity of a graph is using Laplacian eigenvalues. Let $G$ be a connected graph with $k$ vertices. Kelmans and Chelnoknov [14] derived the following formula:

$$\tau(G) = \frac{1}{k}\prod_{i=1}^{k-1}\mu_i. \tag{1.1}$$

Where $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_k = 0$ are the eigenvalues of the Kirchhoff matrix $L$.

Degenerating the graph through successive elimination of contraction of its edges represent the core of another way to compute the complexity of a graph [15,16, 17]. If $G = (V,E)$ is a multigraph with $e \in E$, then $G.e$ is the graph obtained from $G$ by contracting the degree until its endpoints are a single vertex. The formula for computing the number of spanning trees of a multigraph $G$ is given by:

$$\tau(G) = \tau(G-e) + \tau(G.e) \tag{1.2}$$

This formula is beautiful but not practically useful (grows exponentially with the size of the graph- may be as many as $2^{|E(G)|}$ terms. For a summary of further results for calculating umber of spanning trees of graphs, see [18,19,20,21].

## 2. Electrically equivalent transformations

Kirchhoff's motivation was study of electrical networks: an edge- weighted graph can be regarded as an electrical network, where weights are the conductance of the respective edges. The effect conductance between two specific vertices $x, y$ can be written as the quotient of (weighted) number of spanning trees and the (weighted) number of so-called thickets, i.e., spanning forests with exactly two components and property that each of the components contains precisely one of the vertices $x, y$ [22,23,24,25]. In the following, we list the effect of some simple transformations on the number of spanning trees. Let $H$ be an edge weighted graph, $H'$ be the corresponding electrically equivalent graph, $\tau(H)$ denotes the weighted number of spanning trees $H$.

i. Parallel edges: If two parallel edges with conductances $x$ and $y$ in $H$ are merged into a single edge with conductances $x + y$ in $H'$, then $\tau(H') = \tau(H)$.

ii. Serial edges: If two serial edges with conductances $x$ and $y$ in $H$ are merged into a single edge with conductance $\frac{xy}{x+y}$ in $H'$, then $\tau(H') = \frac{1}{x+y}\tau(H)$.

iii. $\Delta - Y$ Transformation: If a triangle with conductances $a, b$ and $c$ in $H$ is changed into an electrically equivalent star graph with conductances $x = \frac{ab+bc+ca}{a}$, $y = \frac{ab+bc+ca}{b}$ and $z = \frac{ab+bc+ca}{c}$ in $H'$, then $\tau(H') = \frac{(ab+bc+ca)^2}{abc}\tau(H)$.

iv. $Y - \Delta$ Transformation: If a star graph with conductances $x, y$ and $z$ in $H$ is changed into an electrically equivalent triangle with conductances $a = \frac{yz}{x+y+z}$, $b = \frac{az}{x+y+z}$ and $c = \frac{xy}{x+y+z}$ in $H'$, then $\tau(H') = \frac{1}{a+b+c}\tau(H)$.

In mathematics one always tries to get new structures from given ones. This also applies to the realm of graphs, where one can generate many new graphs from a given set of graphs.

In this work, we compute the number of spanning trees of four sequences of graphs of average degree four based on Tridiminished icosahedron graph we named it $X_n$, $T_n$ $Y_n$ and $Z_n$ respectively.

## 3. Number of spanning trees in the sequences of $X_n$ graph

Consider the sequence of graphs $X_1, X_2, \ldots, X_n$ constructed as shown in Figure 1.

According to this construction, the number of total vertices $|V(X_n)|$ and edges $|E(X_n)|$ are $|V(X_n)| = 9n - 6$ and $|E(X_n)| = 18n - 15$, $n = 1, 2, \ldots$. The average degree of $X_n$ is in the large $n$ limit which is 4.

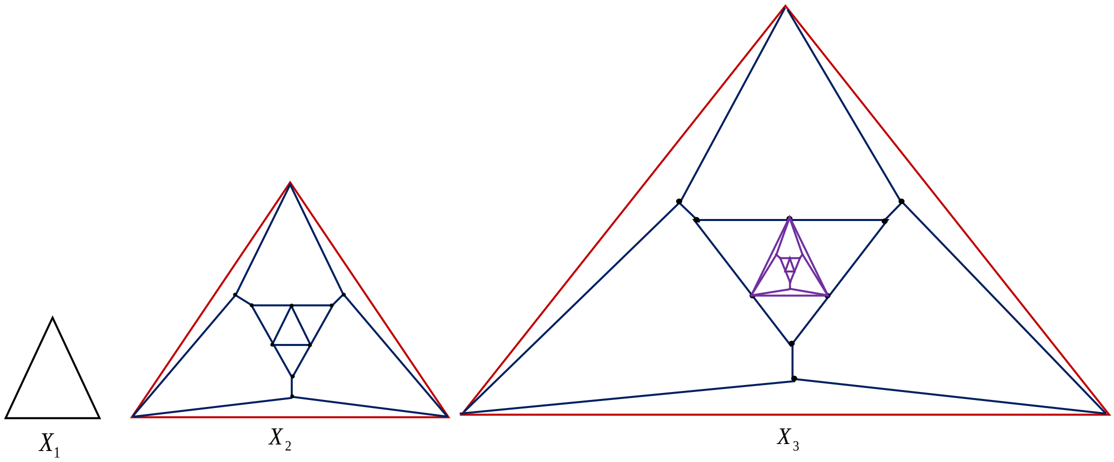
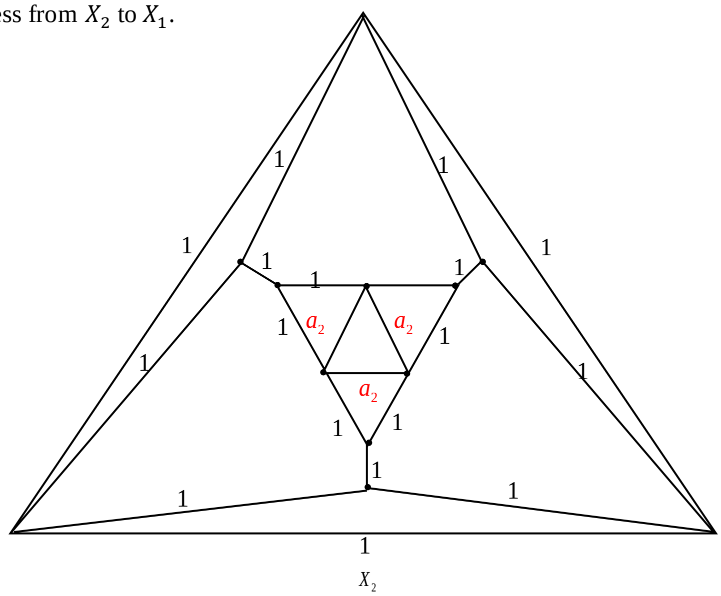


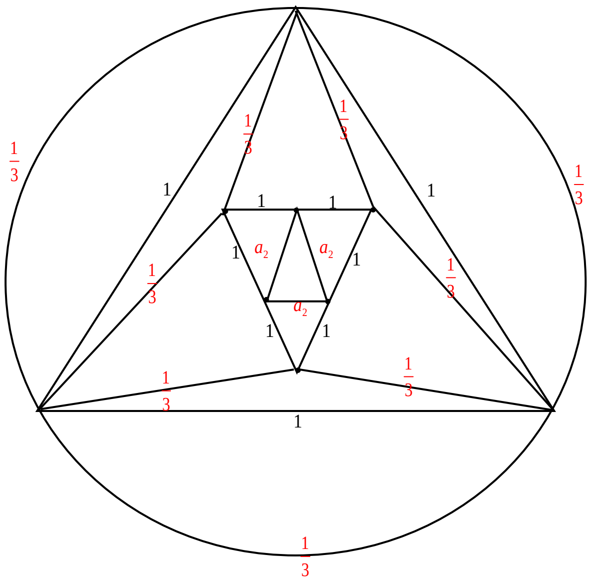

**Fig. (1):** Some sequences of graph $X_n$

**Theorem 1.** For $n \geq 1$, the number of spanning trees in the sequence of the graph $X_n$ is given by

$$\frac{3 \times 4^{n-2}\left((169-27\sqrt{39})(25+4\sqrt{39})^n + (25-4\sqrt{39})^n(169+27\sqrt{39})\right)^2\left(-5(-3+\sqrt{39}) + (63+11\sqrt{39})(1249+200\sqrt{39})^{-1+n}\right)^2}{169\left(30+6(8+\sqrt{39})(1249+200\sqrt{39})^{-1+n}\right)^2}$$

Proof: We use the electrically equivalent transformation to transform $X_i$ to $X_{i-1}$. Fig. (2) illustrates the transformation process from $X_2$ to $X_1$.

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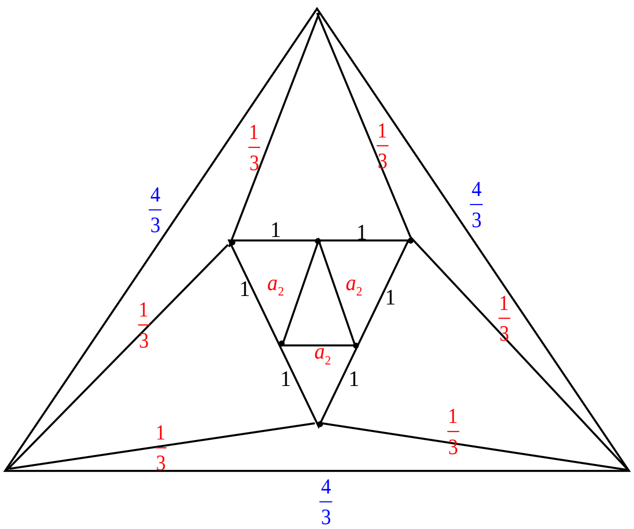
$H_1$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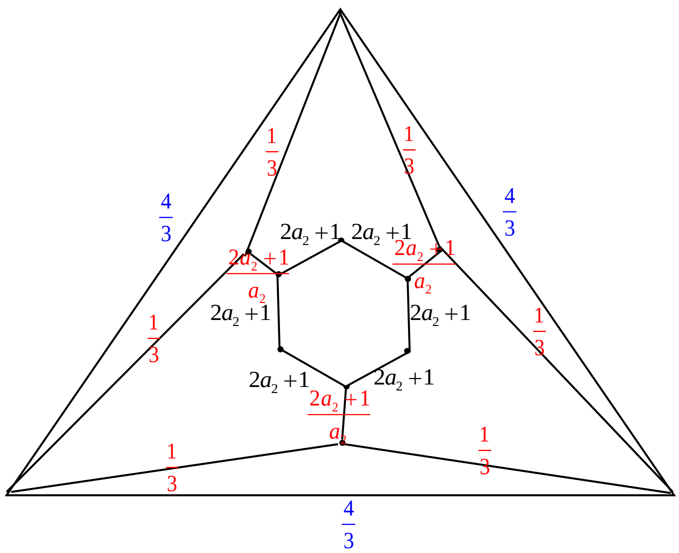
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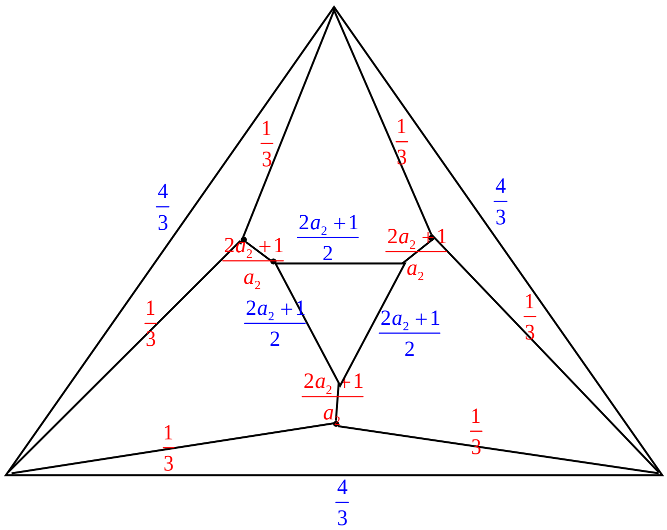
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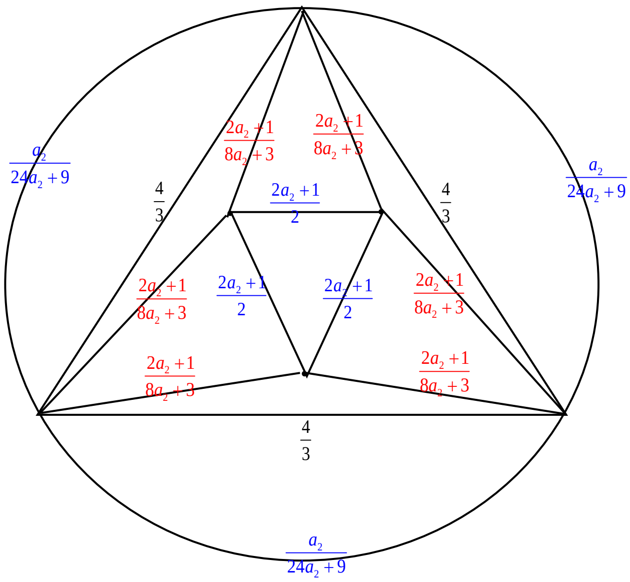
$H_4$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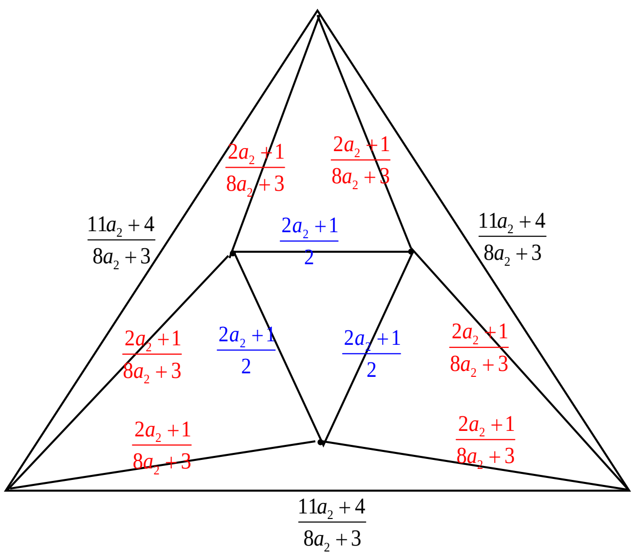
4/3
a_2/(24a_2 + 9)

$H_5$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(2a_2 + 1)/(8a_2 + 3)
(11a_2 + 4)/(8a_2 + 3)

$H_6$

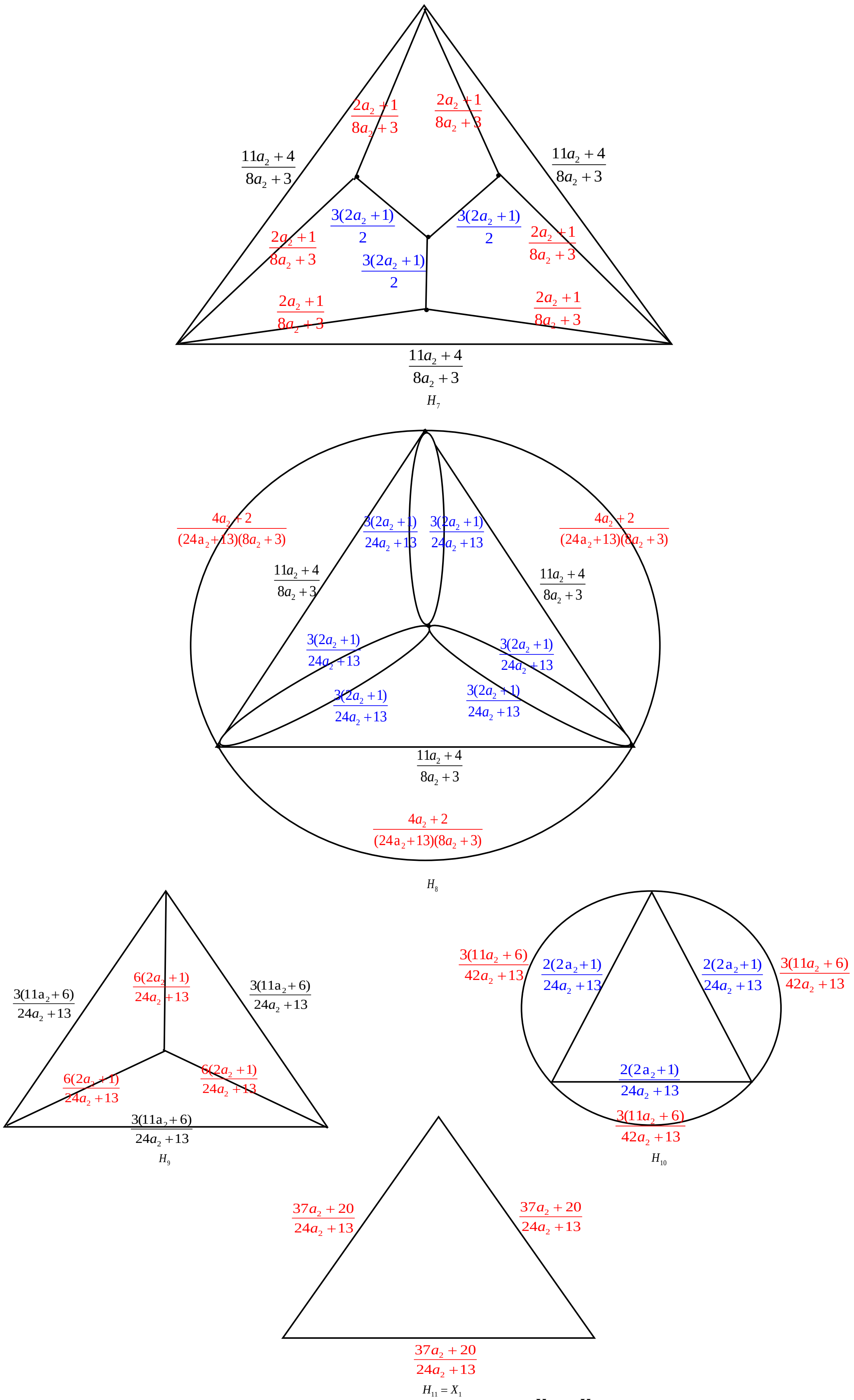



**Fig. (2):** The transformations from $X_2$ to $X_1$.

Using the properties given in section 2 , we have the following the transformations:
$\tau(H_1) = [\frac{1}{3}]^3\tau(X_2), \tau(H_2) = \tau(H_1), \tau(H_3) = [\frac{(3a_2+1)^2}{a_2}]^3\tau(H_2), \tau(H_4) = [\frac{1}{2(2a_2+1)}]^3\tau(H_3), \tau(H_5) = [\frac{3a_2}{8a_2+3}]^3\tau(H_4), \tau(H_6) = \tau(H_5), \tau(H_7) = 9(\frac{2a_2+1}{2})\tau(H_6), \tau(H_8) = [\frac{2(8a_2+3)}{(24a_2+13)(2a_2+1)}]^3\tau(H_6), \tau(H_9) = \tau(H_8), \tau(H_{10}) = \frac{24a_2+13}{18(2a_2+1)}\tau(H_9)$ and $\tau(X_1) = \tau(H_{10})$.

Combining these eleven transformations, we have $\tau(X_2) = 4(24a_2 + 13)^2\tau(X_1)$. (3.1)

Further $\tau(X_n) = \prod_{i=2}^{n} 4(24a_i + 13)^2\tau(X_1) = 3 \times 4^{n-1}a_1^2[\prod_{i=2}^{n}(24a_i + 13)]^2$ (3.2)

Where $a_{i-1} = \frac{37a_i+20}{24a_i+13}, i = 2,3,\dots,n$. Its characteristic equation is $24\lambda^2 - 24\lambda - 20 = 0$, which have two roots

roots $\lambda_1 = \frac{3-\sqrt{39}}{6}$ and $\lambda_2 = \frac{3-\sqrt{39}}{6}$. Subtracting these two roots into both sides of $a_{i-1} = \frac{37a_i+20}{24a_i+13}$, we get

$$a_{i-1} - \frac{3-\sqrt{39}}{6} = \frac{37a_i+20}{24a_i+13} - \frac{3-\sqrt{39}}{6} = (25 + 4\sqrt{39}).\frac{a_i-\frac{3-\sqrt{39}}{6}}{24a_i+13} \quad (3.3)$$

$$a_{i-1} - \frac{3+\sqrt{39}}{6} = \frac{37a_i+20}{24a_i+13} - \frac{3+\sqrt{39}}{6} = (25 - 4\sqrt{39}).\frac{a_i-\frac{3+\sqrt{39}}{6}}{(24a_i+13)} \quad (3.4)$$

Let $b_i = \frac{a_i-\frac{3-\sqrt{39}}{6}}{a_i-\frac{3+\sqrt{39}}{6}}$. Then by Eqs. (3.3) and (3.4), we get $b_{i-1} = (1249 + 200\sqrt{39})b_i$ and $b_i = (1249 + 200\sqrt{39})^{n-i}b_n$.

Therefore

$$a_i = \frac{(1249+200\sqrt{39})b_i)^{n-i}(\frac{3+\sqrt{39}}{6})b_n - \frac{3-\sqrt{39}}{36}}{(1249+200\sqrt{39})b_i)^{n-i}b_n - 1}.$$

Thus

$$a_1 = \frac{(1249+200\sqrt{39})^{n-1}(63+11\sqrt{39})+5(3-\sqrt{39})}{6(1249+200\sqrt{39})^{n-1}(8+\sqrt{39})+30}. \quad (3.5)$$

Using the expression $a_{n-1} = \frac{37a_n+20}{24a_n+13}$ and denoting the coefficients of $37a_n + 20$ and $24a_n + 13$ as $\alpha_n$ and $\beta_n$ we have

$$24a_n + 13 = \alpha_0(37a_n + 20) + \beta_0(24a_n + 13),$$
$$24a_{n-1} + 13 = \frac{\alpha_1(37a_n + 20) + \beta_1(24a_n + 13)}{\alpha_0(37a_n + 20) + \beta_0(24a_n + 13)},$$
$$24a_{n-2} + 13 = \frac{\alpha_2(37a_n + 20) + \beta_2(24a_n + 13)}{\alpha_1(37a_n + 20) + \beta_1(24a_n + 13)},$$
$$\vdots$$
$$24a_{n-i} + 13 = \frac{\alpha_i(37a_n+20)+\beta_i(24a_n+13)}{\alpha_{i-1}(37a_n+20)+\beta_{i-1}(24a_n+13)}, \quad (3.6)$$
$$24a_{n-(i+1)} + 13 = \frac{\alpha_{i+1}(37a_n+20)+\beta_{i+1}(24a_n+13)}{\alpha_i(37a_n+20)+\beta_i(24a_n+13)}, \quad (3.7)$$
$$\vdots$$
$$24a_2 + 13 = \frac{\alpha_{n-2}(37a_n + 20) + \beta_{n-2}(24a_n + 13)}{\alpha_{n-3}(37a_n + 20) + \beta_{n-3}(24a_n + 13)},$$

Substituting Eq. (3.6) into Eq. (3.2), we obtain

$$\tau(X_n) = 3 \times 4^{n-1}a_1^2[\alpha_{n-2}(37a_n + 20) + \beta_{n-2}(24a_n + 13)]^2 \quad (3.8)$$

where $\alpha_0 = 0, \beta_0 = 1$ and $\alpha_1 = 24, \beta_1 = 13$.

By the expression $a_{n-1} = \frac{37a_n+20}{24a_n+13}$ and Eqs. (3.6) and (3.7), we have

$$\alpha_{i+1} = 50\alpha_i - \alpha_{i-1};\ \beta_{i+1} = 50\beta_i - \beta_{i-1} \quad (3.9)$$

The characteristic equation of Eq. (3.9) is $\mu^2 - 50\mu + 1 = 0$ which have two roots $\mu_1 = 25 + 4\sqrt{39}$ and $\mu_2 = 25 - 4\sqrt{39}$.

The general solutions of Eq. (3.9) are $\alpha_i = c_1\mu_1^i + c_2\mu_2^i; \beta_i = d_1\mu_1^i + d_2\mu_2^i$.

Using the initial conditions $\alpha_0 = 0, \beta_0 = 1$ and $\alpha_1 = 24, \beta_1 = 13$, yields

$$\alpha_i = \frac{\sqrt{39}}{13}(25 + 4\sqrt{39})^i - \frac{\sqrt{39}}{13}(25 - 4\sqrt{39})^i; \beta_i = (\frac{39-3\sqrt{39}}{78})(25 + 4\sqrt{39})^i + (\frac{39+3\sqrt{39}}{78})(25 - 4\sqrt{39})^i \quad (3.10)$$

If $a_n = 1$, it means that $X_n$ without any electrically equivalent transformation. Plugging Eq. (3.10) into Eq.(3.8), we have

$$\tau(X_n) = 3 \times 4^{n-1}a_1^2[(\frac{481+77\sqrt{39}}{26})(25 + 4\sqrt{39})^{n-2} + (\frac{481-77\sqrt{39}}{26})(25 - 4\sqrt{39})^{n-2}]^2, n \geq 2. \quad (3.11)$$

When $n = 1$ , $\tau(X_1) = 3$ which satisfies Eq.(3.11). Therefore, the number of spanning trees in the sequence of the graph $X_n$ is given by

$$\tau(X_n) = 3 \times 4^{n-1}a_1^2[(\frac{481+77\sqrt{39}}{26})(25 + 4\sqrt{39})^{n-2} + (\frac{481-77\sqrt{39}}{26})(25 - 4\sqrt{39})^{n-2}]^2, n \geq 1. \quad (3.12)$$

where

$$a_1 = \frac{(1249+200\sqrt{39})^{n-1}(63+11\sqrt{39})+5(3-\sqrt{39})}{6(1249+200\sqrt{39})^{n-1}(8+\sqrt{39})+30}, n \geq 1. \qquad (3.13)$$

Inserting Eq. (3.13) into Eq.(3.12) we obtain the result. □

## 4. Number of spanning trees in the sequences of $T_n$ graph

Consider the sequence of graphs $T_1, T_2, \ldots, T_n$ constructed as shown in Figure 3.
According to this construction, the number of total vertices $|V(T_n)|$ and edges $|E(T_n)|$ are $|V(T_n)| = 9n - 6$ and $|E(T_n)| = 18n - 15$, $n = 1, 2, \ldots$. The average degree of $T_n$ is in the large $n$ limit which is 4.

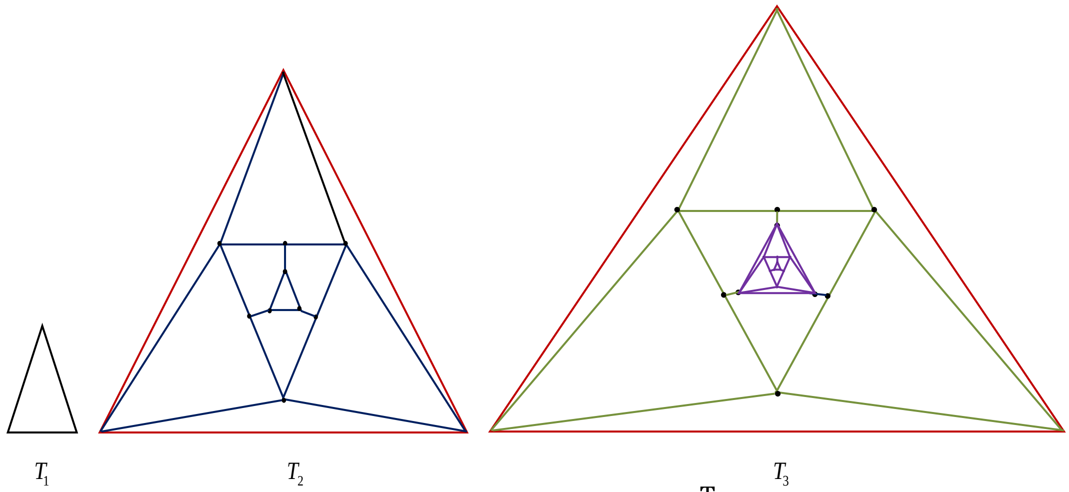
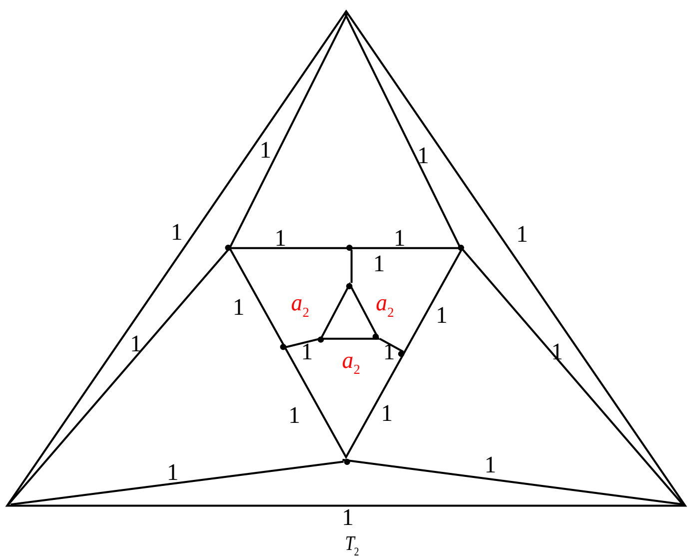


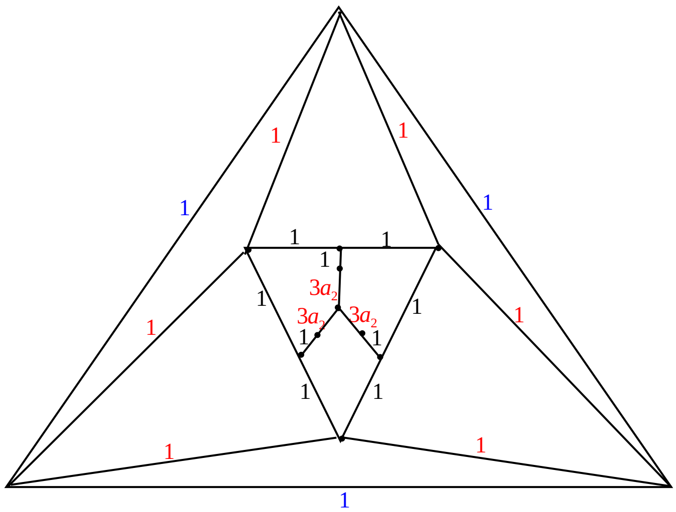

**Fig. (3):** Some sequences of graph $T_n$

**Theorem 2.** For $n \geq 1$, the number of spanning trees in the sequence of $T_n$ graph is given by

$$\frac{(59+\sqrt{3477})^{2n}((1644621-29029\sqrt{3477})(3479+59\sqrt{3477})^{n}-1035\times 2^{n+1}(23180+393\sqrt{3477})+253(3479-59\sqrt{3477})^{n}(48604983+824287\sqrt{3477}))^2}{(16119372(253\times 2^{n}(3479+59\sqrt{3477})+(653+7\sqrt{3477})(3479+59\sqrt{3477})^{n})^2}$$

Proof: We use the electrically equivalent transformation to transform $T_i$ to $T_{i-1}$. Fig. (4) illustrates the transformation process from $T_2$ to $T_1$.

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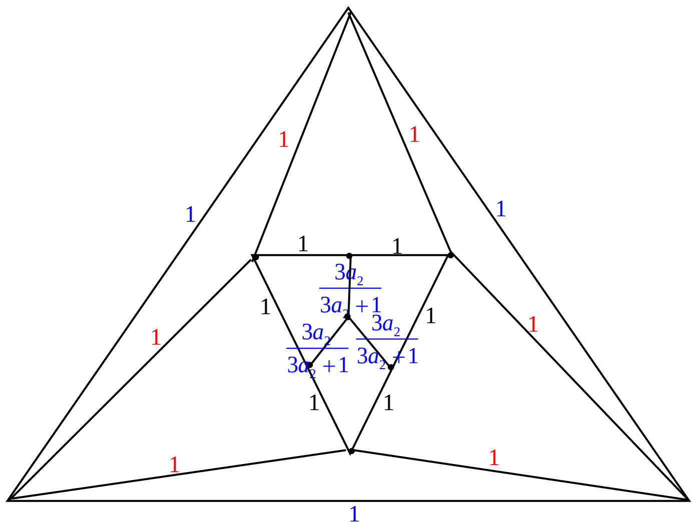
$H_1$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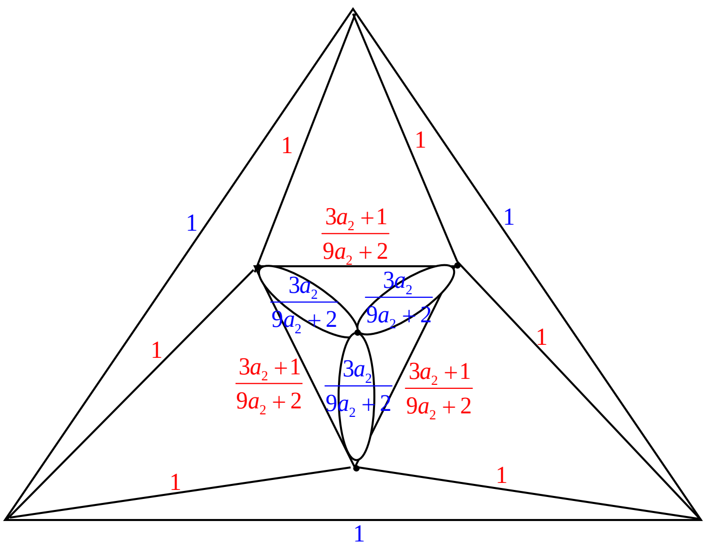
$H_2$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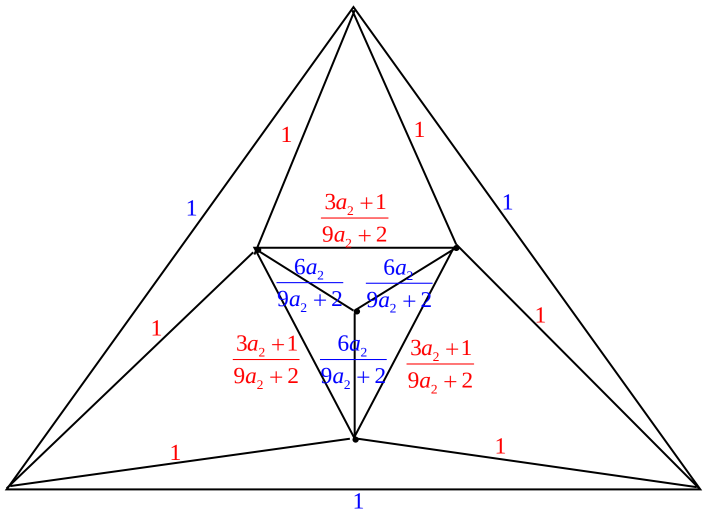
$H_3$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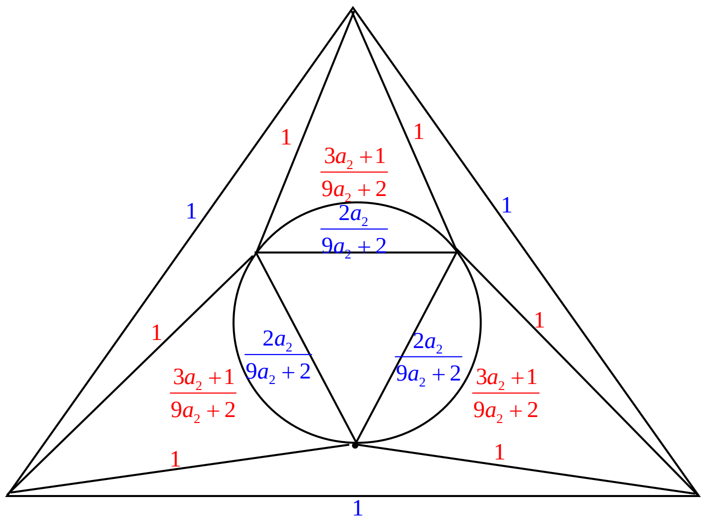
$H_4$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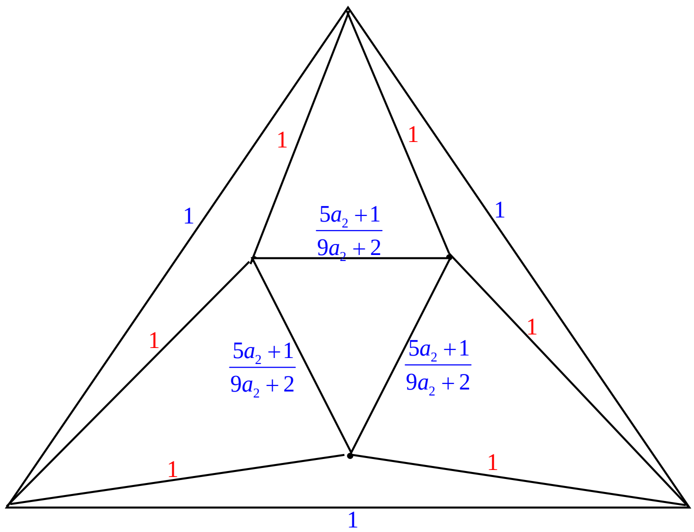
$H_5$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$H_6$

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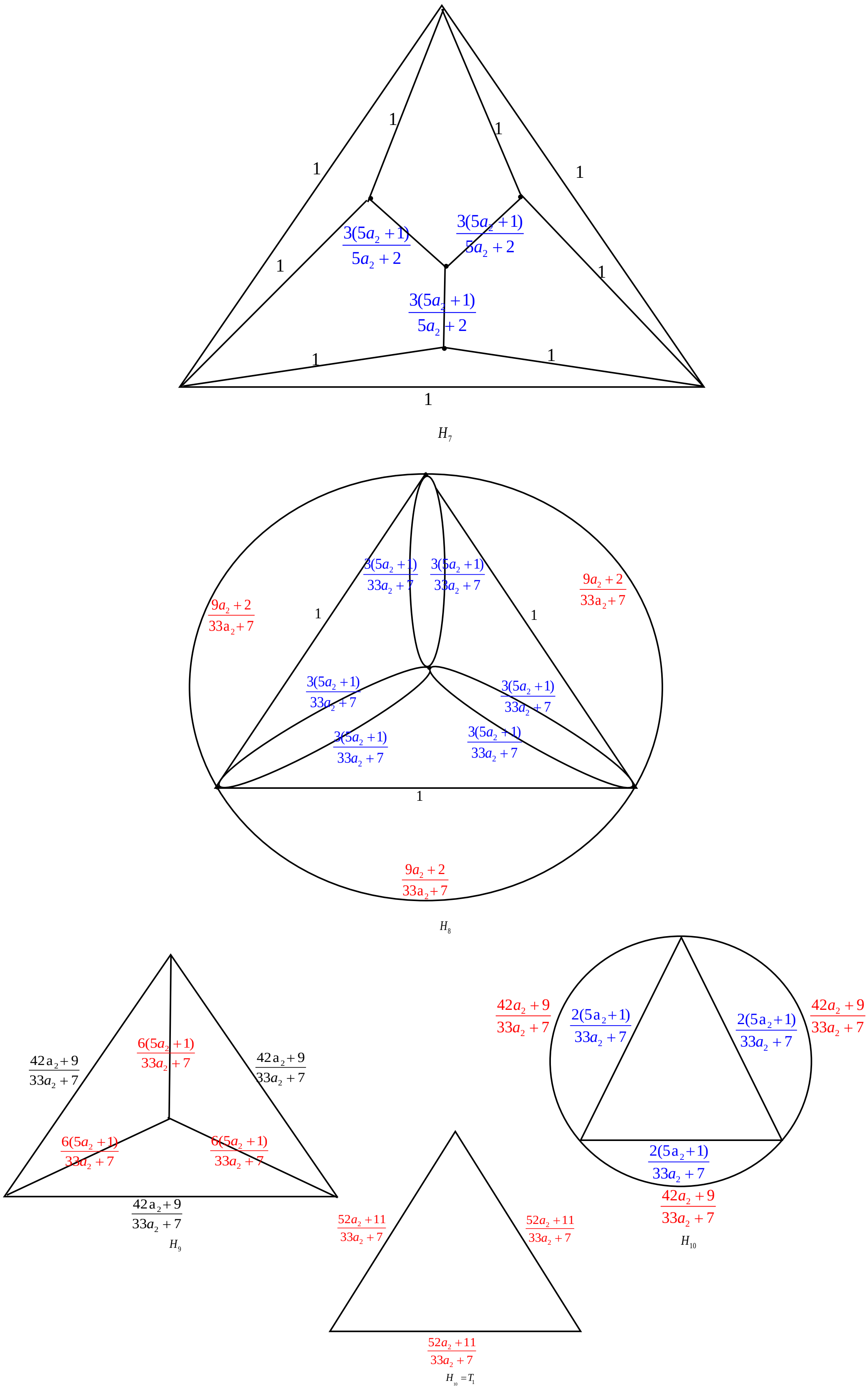



**Fig.(4):** The transformations from $T_2$ to $T_1$

Using the properties given in section 2 , we have the following the transformations:
$\tau(H_1) = 9a_2\tau(T_2), \tau(H_2) = [\frac{1}{3a_2+1}]^3\tau(H_1), \tau(H_3) = [\frac{3a_2+1}{9a_2+2}]^3\tau(H_2), \tau(H_4) = \tau(H_3), \tau(H_5) = (\frac{9a_2+2}{18a_2})\tau(H_4)$, $\tau(H_6) = \tau(H_5), \tau(H_7) = \frac{9(5a_2+1)}{9a_2+2}\tau(H_6), \tau(H_8) = [\frac{9a_2+2}{33a_2+7}]^3\tau(H_7), \tau(H_9) = \tau(H_8), \tau(H_{10}) = [\frac{33a_2+7}{18(5a_2+1)}]^3\tau(H_9)$ and $\tau(T_1) = \tau(H_{10})$. Combining these eleven transformations, we have

$$\tau(T_2) = 4(33a_2 + 7)^2\tau(T_1). \quad (4.1)$$

Further $\tau(T_n) = \prod_{i=2}^{n} 4(33a_i + 7)^2\tau(T_1) = 3 \times 4^{n-1}a_1^2[\prod_{i=2}^{n}(33a_i + 7)]^2$ (4.2)

where $a_{i-1} = \frac{52a_i+11}{33a_i+7}, i = 2,3,\ldots,n$.

Its characteristic equation is $33\lambda^2 - 45\lambda - 11 = 0$ which have two roots $\lambda_1 = \frac{45-\sqrt{3477}}{66}$ and $\lambda_2 = \frac{45+\sqrt{3477}}{66}$.
Subtracting these two roots into both sides of $a_{i-1} = \frac{52a_i+11}{33a_i+7}$, we get

$$a_{i-1} - \frac{45-\sqrt{3477}}{66} = \frac{52a_i+11}{33a_i+7} - \frac{45-\sqrt{3477}}{66} = (59 + \sqrt{3477}).\frac{a_i-(\frac{45-\sqrt{3477}}{66})}{2(33a_i+7)} \quad (4.3)$$

$$a_{i-1} - \frac{45+\sqrt{3477}}{66} = \frac{52a_i+11}{33a_i+7} - \frac{45+\sqrt{3477}}{66} = (59 - \sqrt{3477}).\frac{a_i-(\frac{45+\sqrt{3477}}{66})}{2(33a_i+7)} \quad (4.4)$$

Let $b_i = \frac{a_i-\frac{45-\sqrt{3477}}{66}}{a_i-\frac{45+\sqrt{3477}}{66}}$. Then by Eqs. (4.3) and (4.4), we get $b_{i-1} = (\frac{3479+59\sqrt{3477}}{2})b_i$ and $b_i = (\frac{3479+59\sqrt{3477}}{2})^{n-i}b_n$.
Therefore

$$a_i = \frac{(\frac{3479+59\sqrt{3477}}{2})^{n-i}(\frac{45+\sqrt{3477}}{66})b_n - \frac{45-\sqrt{3477}}{66}}{(\frac{3479+59\sqrt{3477}}{2})^{n-i}b_n - 1}.$$ Thus

$$a_1 = \frac{(\frac{3479+59\sqrt{3477}}{2})^{n-1}(\frac{111+2\sqrt{3477}}{69}) + \frac{45-\sqrt{3477}}{66}}{(\frac{3479+59\sqrt{3477}}{2})^{n-1}(\frac{653+7\sqrt{3477}}{506})+1}. \quad (4.5)$$

Using the expression $a_{n-1} = \frac{52a_n+11}{33a_n+7}$ and denoting the coefficients of $52a_n + 11$ and $33a_n + 7$ as $\alpha_n$ and $\beta_n$, we have

$$33a_n + 7 = \alpha_0(52a_n + 11) + \beta_0(33a_n + 7),$$
$$33a_{n-1} + 7 = \frac{\alpha_1(52a_n + 11) + \beta_1(33a_n + 7)}{\alpha_0(52a_n + 11) + \beta_0(33a_n + 7)},$$
$$33a_{n-2} + 7 = \frac{\alpha_2(52a_n + 11) + \beta_2(33a_n + 7)}{\alpha_1(52a_n + 11) + \beta_1(33a_n + 7)},$$
$$\vdots$$
$$33a_{n-i} + 7 = \frac{\alpha_i(52a_n+11)+\beta_i(33a_n+7)}{\alpha_{i-1}(52a_n+11)+\beta_{i-1}(33a_n+7)}, \quad (4.6)$$
$$33a_{n-(i+1)} + 7 = \frac{\alpha_{i+1}(52a_n+11)+\beta_{i+1}(33a_n+7)}{\alpha_i(52a_n+11)+\beta_i(33a_n+7)}, \quad (4.7)$$
$$\vdots$$
$$33a_2 + 7 = \frac{\alpha_{n-2}(52a_n + 11) + \beta_{n-2}(33a_n + 7)}{\alpha_{n-3}(52a_n + 11) + \beta_{n-3}(33a_n + 7)},$$

Substituting Eq.(4.6) into Eq.(4.2), we obtain

$$\tau(T_n) = 3 \times 4^{n-1}a_1^2[\alpha_{n-2}(52a_n + 11) + \beta_{n-2}(33a_n + 7)]^2 \quad (4.8)$$

where $\alpha_0 = 0, \beta_0 = 1$ and $\alpha_1 = 33, \beta_1 = 7$. By the expression $a_{n-1} = \frac{52a_n+11}{33a_n+7}$ and Eqs. (4.6) and (4.7), we have

$$\alpha_{i+1} = 59\alpha_i - \alpha_{i-1}; \beta_{i+1} = 59\beta_i - \beta_{i-1} \quad (4.9)$$

The characteristic equation of Eq. (4.9) is $\mu^2 - 59\mu + 1 = 0$ which have two roots $\mu_1 = \frac{59+\sqrt{3477}}{2}$ and $\mu_2 = \frac{59-\sqrt{3477}}{2}$.

The general solution of Eq. (4.9) are $\alpha_i = c_1\mu_1^i + c_2\mu_2^i; \beta_i = d_1\mu_1^i + d_2\mu_2^i$.
Using the initial conditions $\alpha_0 = 0, \beta_0 = 1$ and $\alpha_1 = 33, \beta_1 = 7$, yields

$$\alpha_i = \frac{11\sqrt{3477}}{1159}(\frac{59 + \sqrt{3477}}{2})^i - \frac{11\sqrt{3477}}{1159}(\frac{59 - \sqrt{3477}}{2})^i;$$
$$\beta_i = (\frac{1159-15\sqrt{3477}}{2318})(\frac{59+\sqrt{3477}}{2})^i + (\frac{1159+15\sqrt{3477}}{2318})(\frac{59-\sqrt{3477}}{2})^i \quad (4.10)$$

If $a_n = 1$, it means that $T_n$ without any electrically equivalent transformation. Plugging Eq. (4.10) into Eq.(4.8), we have

$$\tau(T_n) = 3 \times 4^{n-1}a_1^2[(\frac{23180+393\sqrt{3477}}{1159})(\frac{59+\sqrt{3477}}{2})^{n-2} + (\frac{23180-393\sqrt{3477}}{1159})(\frac{59-\sqrt{3477}}{2})^{n-2}]^2, n \geq 2. \quad (4.11)$$

When $n = 1$ , $\tau(T_1) = 3$ which satisfies Eq.(4.11). Therefore, the number of spanning trees in the sequence of Tridiminished icosahedron graph is given by

$$\tau(T_n) = 3 \times 4^{n-1} a_1^2 [(\frac{23180+393\sqrt{3477}}{1159})(\frac{59+\sqrt{3477}}{2})^{n-2} + (\frac{23180-393\sqrt{3477}}{1159})(\frac{59-\sqrt{3477}}{2})^{n-2}]^2, n \geq 1. \quad (4.12)$$

where

$$a_1 = \frac{(\frac{3479+59\sqrt{3477}}{2})^{n-1}(\frac{111+2\sqrt{3477}}{69})+(\frac{45-\sqrt{3477}}{66})}{(\frac{3479+59\sqrt{3477}}{2})^{n-1}(\frac{653+7\sqrt{3477}}{506})+1}, n \geq 1. \quad (4.13)$$

Inserting Eq.(4.13) into Eq.(4.12) we obtain the result. □

## 5. Number of spanning trees in the sequences of $Y_n$ graph

Consider the sequence of graphs $Y_1, Y, \ldots, Y_n$ constructed as shown in Figure 5.
According to this construction, the number of total vertices $|V(Y_n)|$ and edges $|E(Y_n)|$ are $|V(Y_n)| = 9n - 6$ and $|E(Y_n)| = 18n - 15$, $n = 1, 2, \ldots$. The average degree of $Y_n$ is in the large $n$ limit which is 4.

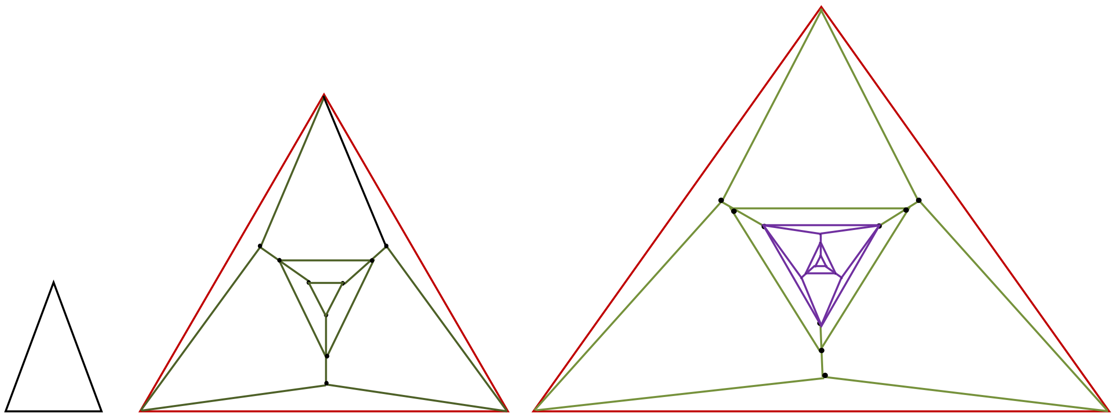
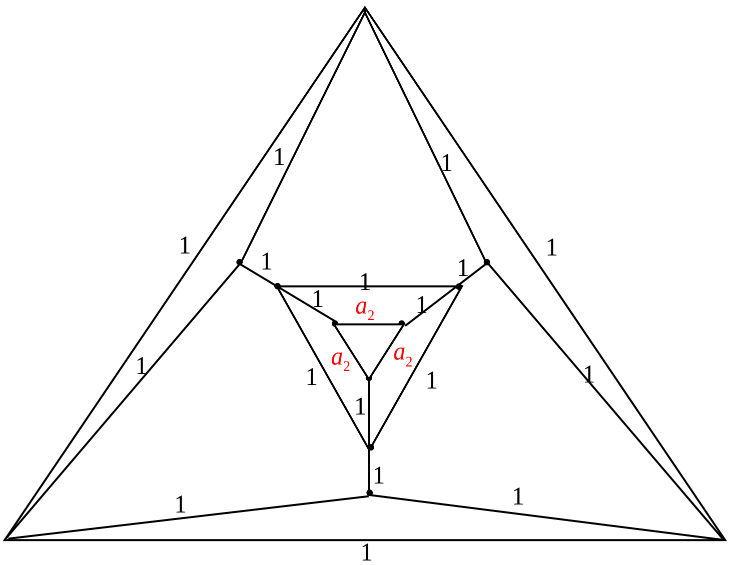


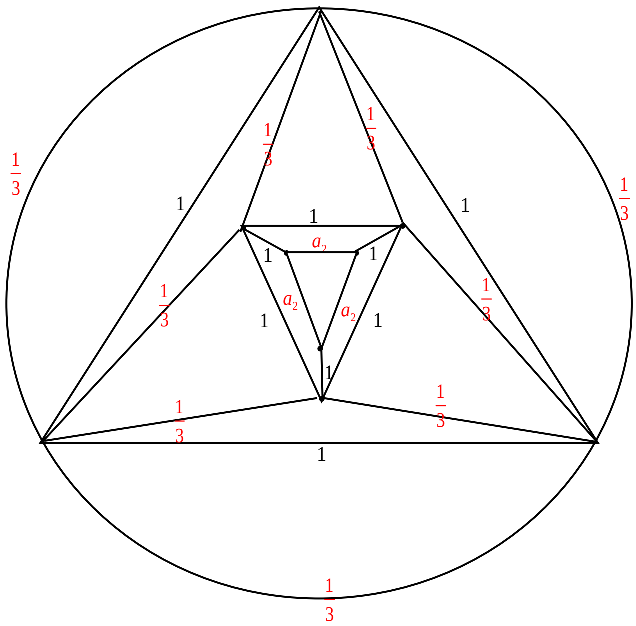

**Fig. (5):** Some sequences of $Y_n$

**Theorem 3.** For $n \geq 1$, the number of spanning trees in the sequence of $Z_n$ is given by

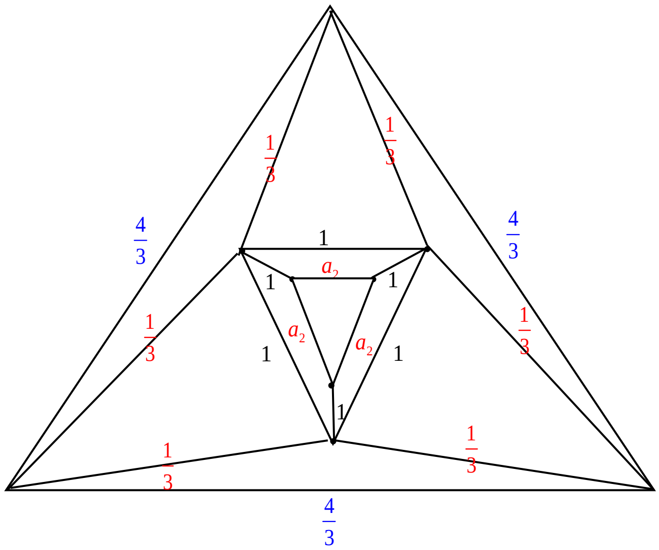
$$\frac{3 \times 2^{n-3}\left(-29(-27+\sqrt{1443})+(2103+59\sqrt{1443})(2887+76\sqrt{1443})^{n-1}\right)^2\left((11063-291\sqrt{1443})(38+\sqrt{1443})^n+(38-\sqrt{1443})^n(11063+291\sqrt{1443})\right)^2}{231361\left(1218+6(278+5\sqrt{1443})(2887+76\sqrt{1443})^{n-1}\right)^2}$$

Proof: We use the electrically equivalent transformation to transform $Y_i$ to $Y_{i-1}$. Fig.(6) illustrates the transformation process from $Y_2$ to $Y_1$.

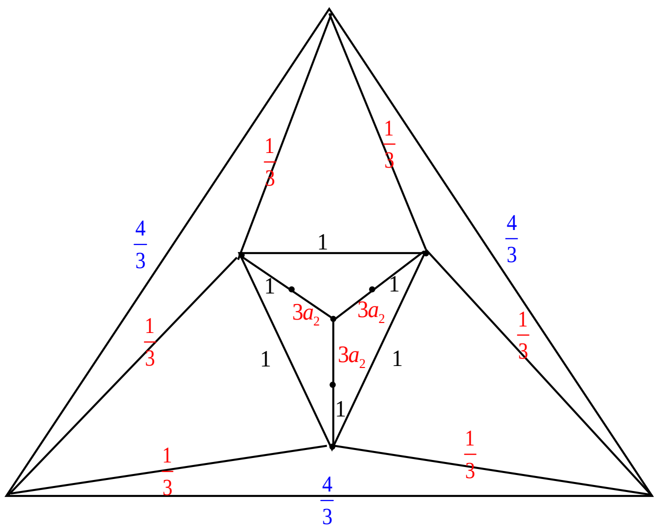




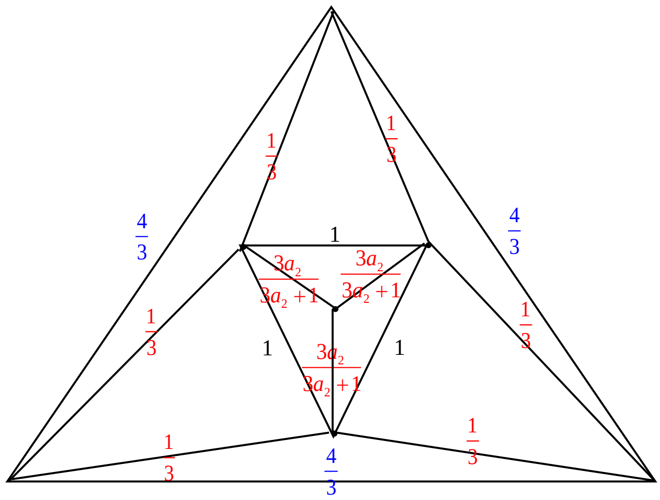

$H_1$




$H_2$




$H_3$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$\frac{1}{3}$
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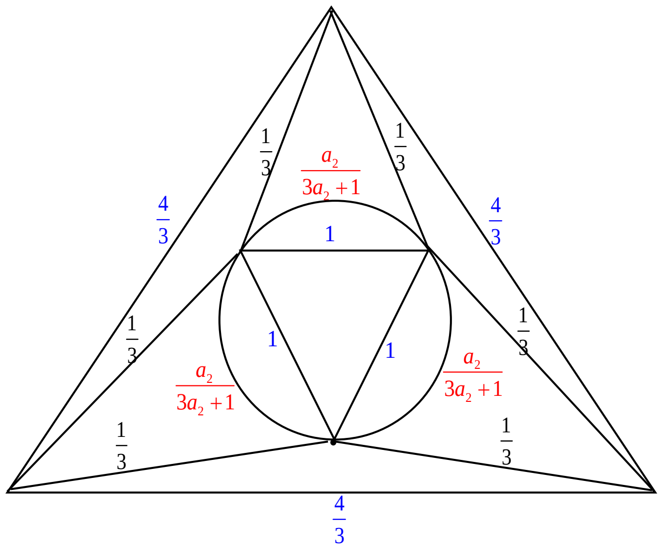
1
$\frac{1}{3}$
$\frac{1}{3}$
$\frac{4}{3}$

$H_4$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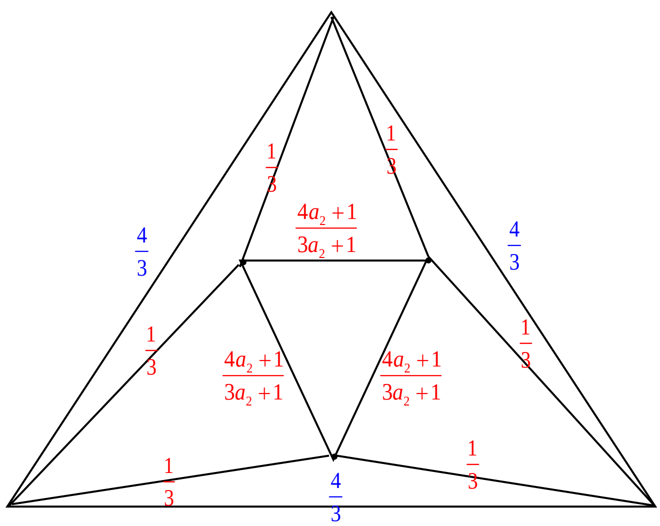
$\frac{4}{3}$

$H_5$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$\frac{4a_2+1}{3a_2+1}$
$\frac{4a_2+1}{3a_2+1}$
$\frac{1}{3}$
$\frac{1}{3}$
$\frac{4}{3}$

$H_6$

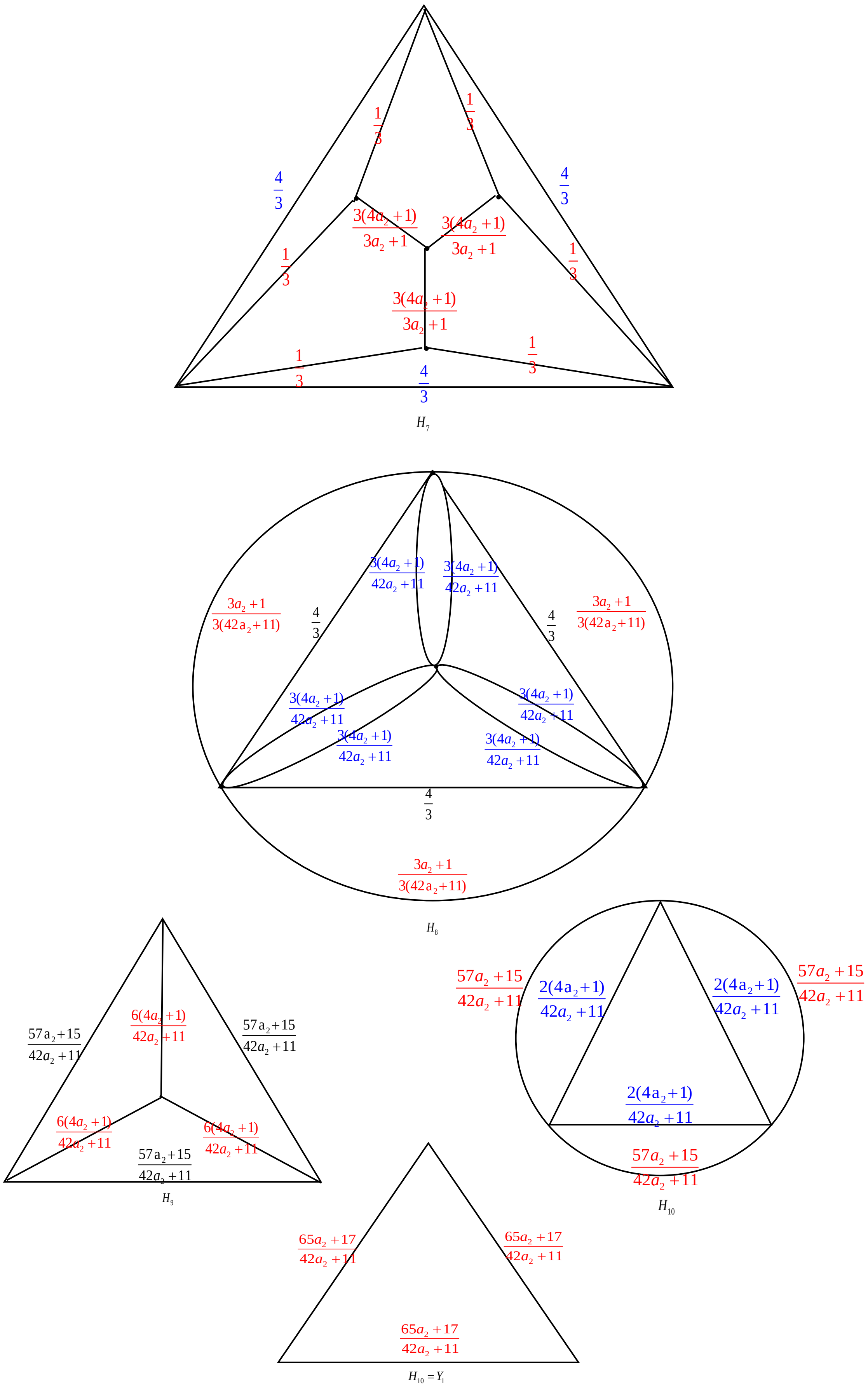



**Fig.(6):** The transformations from $Y_2$ to $Y_1$

Using the properties given in section 2, we have the following the transformations:

$$\tau(H_1) = [\frac{1}{3}]^3\tau(Y_2), \tau(H_2) = \tau(H_1), \tau(H_3) = 9a_2\tau(H_2), \tau(H_4) = (\frac{1}{3a_2+1})^2\tau(H_3),$$

$\tau(H_5) = \frac{3a_2+1}{9a_2}\tau(H_4), \tau(H_6) = \tau(H_5), \tau(H_7) = 9(\frac{4a_2+1}{3a_2+1})\tau(H_6), \tau(H_8) = [\frac{3(3a_2+1)}{42a_2+11}]^3\tau(H_7), \tau(H_9) = \tau(H_8)$, $\tau(H_{10}) = \frac{42a_2+11}{18(4a_2+1)}\tau(H_9)$ and $\tau(Y_1) = \tau(H_{10})$. Combining these eleven transformations, we have

$$\tau(Y_2) = 2(42a_2 + 11)^2\tau(Y_1). \quad (5.1)$$

Further $\tau(Y_n) = \prod_{i=2}^{n} 2(42a_i + 11)^2\tau(Y_1) = 3 \times 2^{n-1}a_1^2[\prod_{i=2}^{n}(42a_i + 11)]^2$ (5.2)

where $a_{i-1} = \frac{65a_i+17}{42a_i+11}, i = 2,3,\ldots,n$. Its characteristic equation is $42\lambda^2 - 54\lambda - 17 = 0$ which have two roots

$\lambda_1 = \frac{27-\sqrt{1443}}{42}$ and $\lambda_2 = \frac{27+\sqrt{1443}}{42}$.

Subtracting these two roots into both sides of $a_{i-1} = \frac{65a_i+17}{42a_i+11}$, we get

$$a_{i-1} - \frac{27-\sqrt{1443}}{42} = \frac{65a_i+17}{42a_i+11} - \frac{27-\sqrt{1443}}{42} = (38+\sqrt{1443}).\frac{a_i - \frac{27-\sqrt{1443}}{42}}{42a_i+11} \quad (5.3)$$

$$a_{i-1} - \frac{27+\sqrt{1443}}{42} = \frac{65a_i+17}{42a_i+11} - \frac{27+\sqrt{1443}}{42} = (38-\sqrt{1443}).\frac{a_i - \frac{27+\sqrt{1443}}{42}}{42a_i+11} \quad (5.4)$$

Let $b_i = \frac{a_i - \frac{27-\sqrt{1443}}{42}}{a_i - \frac{27+\sqrt{1443}}{42}}$. Then by Eqs. (5.3) and (5.4), we get $b_{i-1} = (2887+76\sqrt{1443})b_i$ and $b_i = (2887 + 76\sqrt{1443})^{n-i}b_n$. Therefore.

$$a_i = \frac{(2887+76\sqrt{1443})^{n-i}(\frac{27+\sqrt{1443}}{42})b_n - \frac{27-\sqrt{1443}}{42}}{(2887+76\sqrt{1443})^{n-i}b_n - 1}.$$

Thus

$$a_1 = \frac{(2887+76\sqrt{1443})^{n-1}(2103+59\sqrt{1443})+29(27-\sqrt{1443})}{6(2887+76\sqrt{1443})^{n-1}(278+5\sqrt{1443})+1218}. \quad (5.5)$$

Using the expression $a_{n-1} = \frac{65a_n+17}{42a_n+11}$ and denoting the coefficients of $65a_n + 17$ and $42a_n + 11$ as $\alpha_n$ and $\beta_n$, we have

$$42a_n + 11 = \alpha_0(65a_n + 17) + \beta_0(42a_n + 11),$$

$$42a_{n-1} + 11 = \frac{\alpha_1(65a_n + 17) + \beta_1(42a_n + 11)}{\alpha_0(65a_n + 17) + \beta_0(42a_n + 11)},$$

$$42a_{n-2} + 11 = \frac{\alpha_2(65a_n + 17) + \beta_2(42a_n + 11)}{\alpha_1(65a_n + 17) + \beta_1(42a_n + 11)},$$

$$\vdots$$

$$42a_{n-i} + 11 = \frac{\alpha_i(65a_n+17)+\beta_i(42a_n+11)}{\alpha_{i-1}(65a_n+17)+\beta_{i-1}(42a_n+11)}, \quad (5.6)$$

$$42a_{n-(i+1)} + 11 = \frac{\alpha_{i+1}(65a_n+17)+\beta_{i+1}(42a_n+11)}{\alpha_i(65a_n+17)+\beta_i(42a_n+11)}, \quad (5.7)$$

$$\vdots$$

$$42a_2 + 11 = \frac{\alpha_{n-2}(65a_n + 17) + \beta_{n-2}(42a_n + 11)}{\alpha_{n-3}(65a_n + 17) + \beta_{n-3}(42a_n + 11)},$$

Substituting Eq.(5.6) into Eq.(5.2), we obtain

$$\tau(Y_n) = 3 \times 2^{n-1}a_1^2[\alpha_{n-2}(65a_n + 17) + \beta_{n-2}(42a_n + 11)]^2 \quad (5.8)$$

where $\alpha_0 = 0, \beta_0 = 1$ and $\alpha_1 = 42, \beta_1 = 11$. By the expression $a_{n-1} = \frac{65a_n+17}{42a_n+11}$ and Eqs. (5.6) and (5.7), we have

$$\alpha_{i+1} = 76\alpha_i - \alpha_{i-1}; \beta_{i+1} = 76\beta_i - \beta_{i-1} \quad (5.9)$$

The characteristic equation of Eq. (5.9) is $\mu^2 - 76\mu + 1 = 0$ which have two roots $\mu_1 = 38 + \sqrt{1443}$ and $\mu_2 = 38 - \sqrt{1443}$ . The general solutions of Eq.(5.9) are $\alpha_i = c_1\mu_1^i + c_2\mu_2^i; \beta_i = d_1\mu_1^i + d_2\mu_2^i$.

Substituting the initial conditions $\alpha_0 = 0, \beta_0 = 1$ and $\alpha_1 = 42, \beta_1 = 11$, yields

$$\alpha_i = \frac{7\sqrt{1443}}{481}(38+\sqrt{1443})^i - \frac{7\sqrt{1443}}{481}(38-\sqrt{1443})^i;$$

$$\beta_i = (\frac{481-9\sqrt{1443}}{962})(38+\sqrt{1443})^i + (\frac{481+9\sqrt{1443}}{962})(38-\sqrt{1443})^i \quad (5.10)$$

If $a_n = 1$, it means that $Y_n$ without any electrically equivalent transformation. Plugging Eq. (5.10) into Eq.(5.8), we have

$$\tau(Y_n) = 3 \times 2^{n-1}a_1^2[(\frac{25493+671\sqrt{1443}}{962})(38+\sqrt{1443})^{n-2} + (\frac{25493-671\sqrt{1443}}{962})(38-\sqrt{1443})^{n-2}]^2, n \geq 2. \quad (5.11)$$

When $n = 1$ , $\tau(Y_1) = 3$ which satisfies Eq. (5.11). Therefore, the number of spanning trees in the sequence of $Y_n$ graph is given by

$$\tau(Y_n) = 3 \times 2^{n-1} a_1^2 [(\frac{25493+671\sqrt{1443}}{962})(38+\sqrt{1443})^{n-2} + (\frac{25493-671\sqrt{1443}}{962})(38-\sqrt{1443})^{n-2}]^2, n \geq 1 \quad (5.12)$$

Where

$$a_1 = \frac{(2887+76\sqrt{1443})^{n-1}(2103+59\sqrt{1443})+29(27-\sqrt{1443})}{6(2887+76\sqrt{1443})^{n-1}(278+5\sqrt{1443})+1218}, n \geq 1. \quad (5.13)$$

Inserting Eq. (5.13) into Eq.(5.12) we obtain the result. □

## 6. Number of spanning trees in the sequences of $Z_n$ graph

Consider the sequence of graphs $Z_1, Z_2, \ldots, Z_n$ constructed as shown in Figure 7.

According to this construction, the number of total vertices $|V(Z_n)|$ and edges $|E(Z_n)|$ are $|V(Z_n)| = 9n - 6$ and $|E(Z_n)| = 21n - 18$, $n = 1, 2, \ldots$. The average degree of $Z_n$ is in the large $n$ limit which is $\frac{14}{3}$.

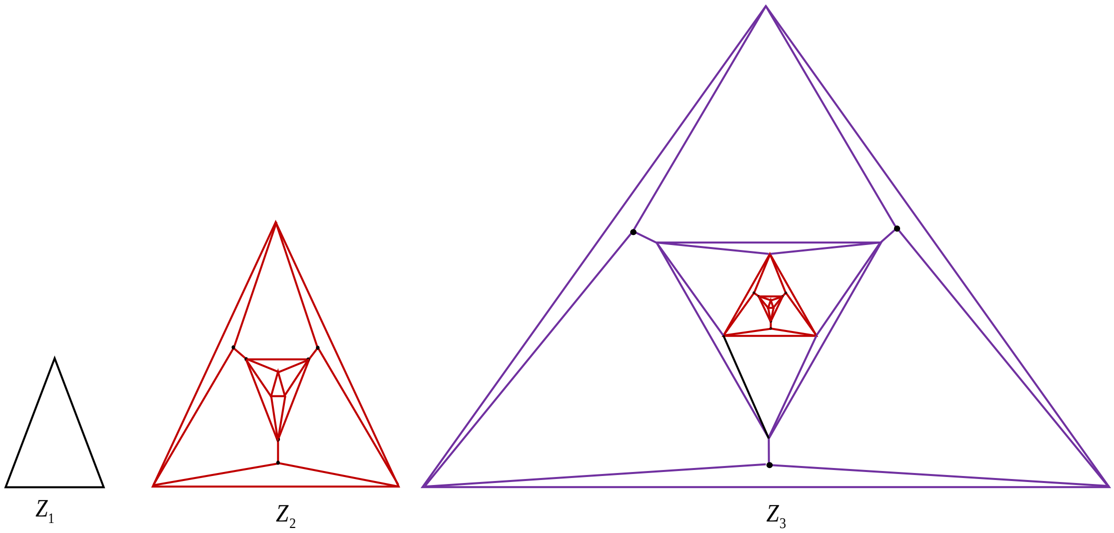
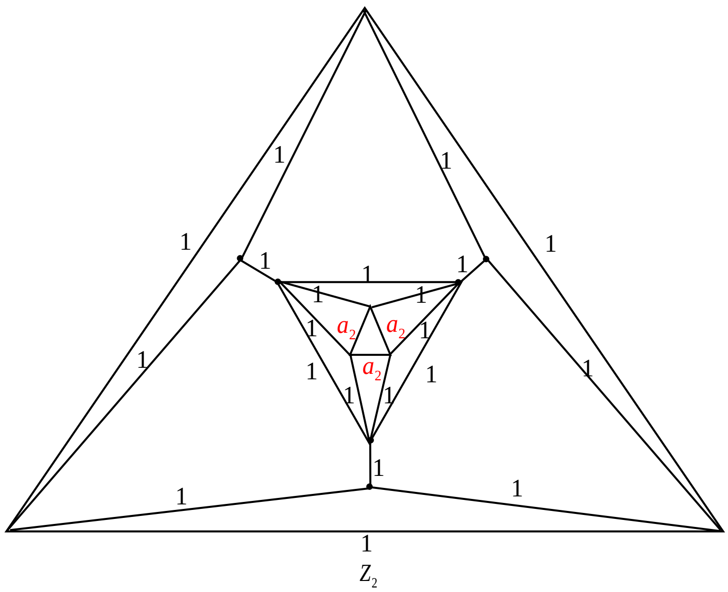


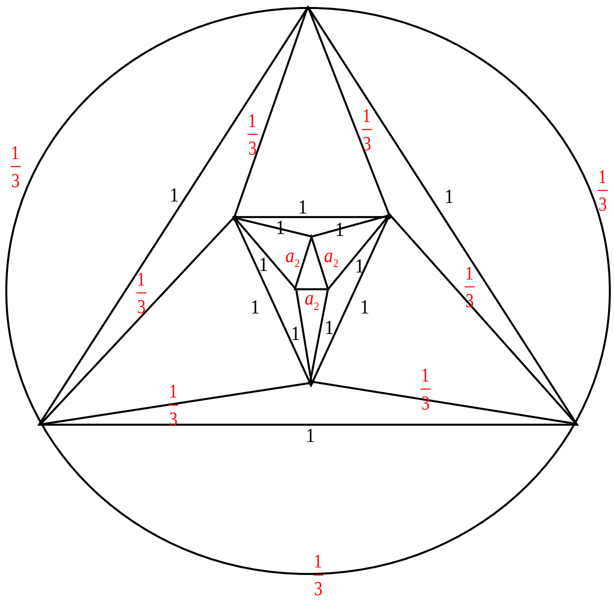

**Fig. (7):** Some sequences of $Z_n$

**Theorem 4.** For $n \geq 1$, the number of spanning trees in the sequence of $Z_n$ is given by

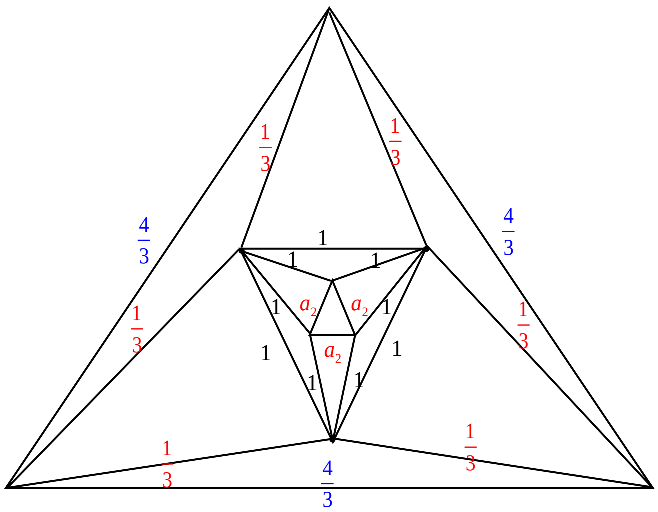
$$\frac{3\times 4^{n-4}\left((784-171\sqrt{21})(55+12\sqrt{21})^n+(55-12\sqrt{21})^n(784+171\sqrt{21})\right)^2\left(20(-2+\sqrt{21})-4(46+11\sqrt{21})(6049+1320\sqrt{21})^{n-1}\right)^2}{49\left(85+(139+24\sqrt{21})(6049+1320\sqrt{21})^{n-1}\right)^2}$$

Proof: We use the electrically equivalent transformation to transform $Z_i$ to $Z_{i-1}$. Fig. (7) illustrates the transformation process from $Z_2$ to $Z_1$.

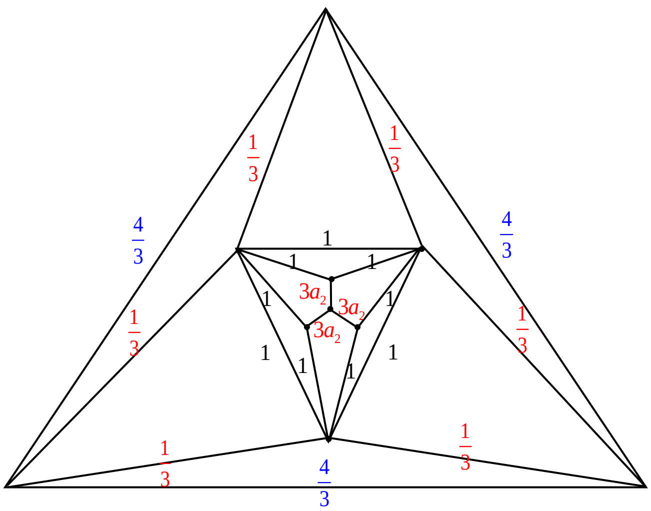




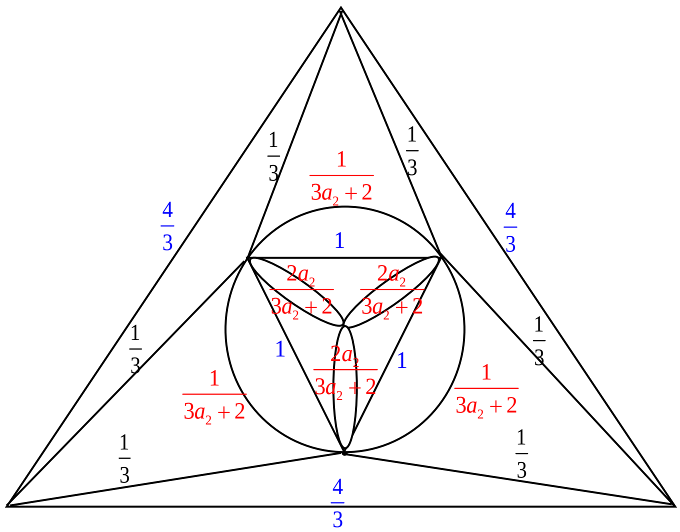

$H_1$




$H_2$




$H_3$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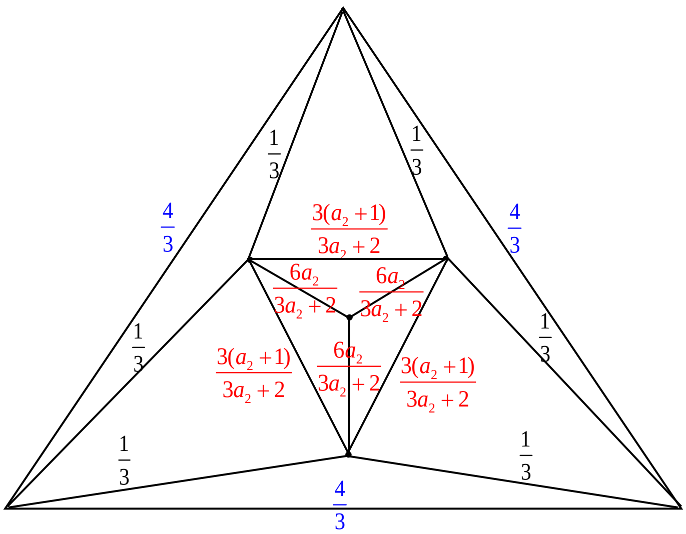
$\frac{1}{3a_2+2}$
$\frac{4}{3}$

$H_4$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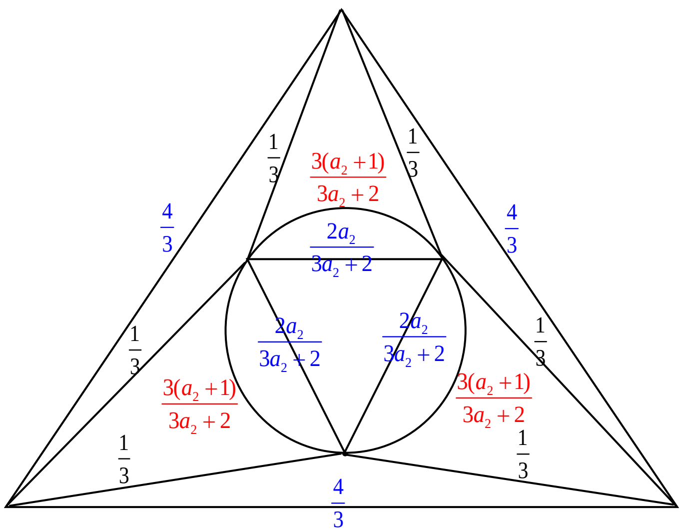
$H_5$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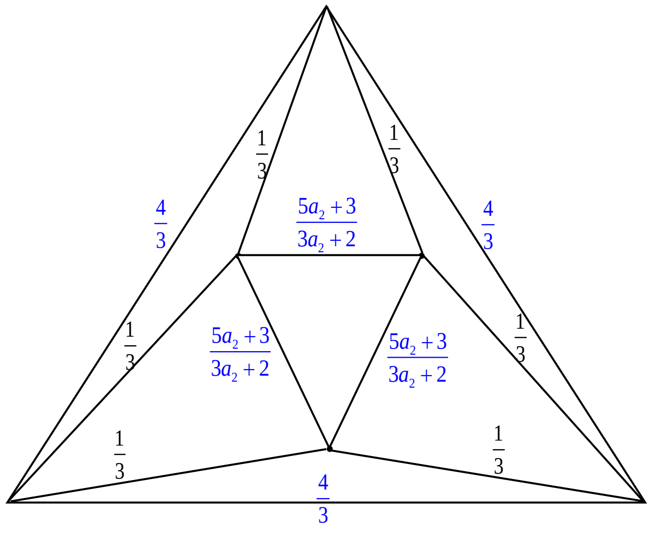
$H_6$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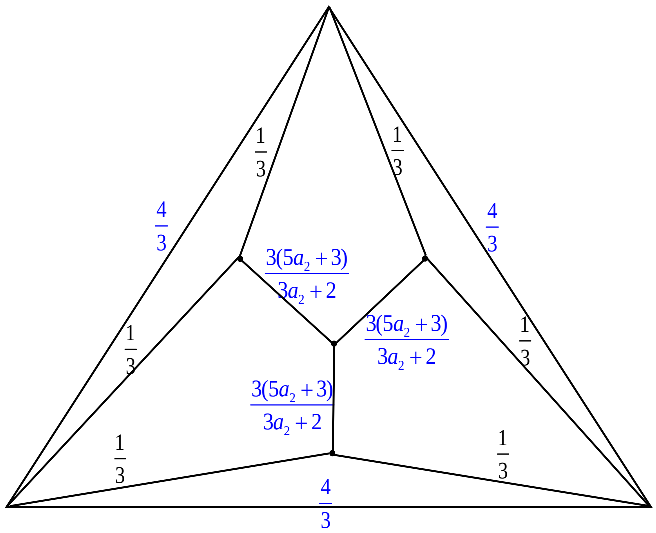
$H_7$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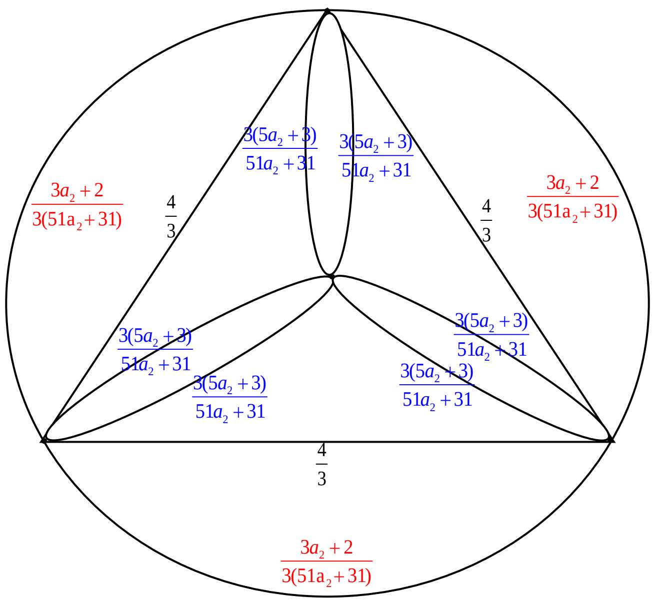
3(5a₂+3) / 3a₂+2
1/3
3(5a₂+3) / 3a₂+2
1/3
1/3
4/3

$H_8$


3(5a₂+3) / 51a₂+31
3(5a₂+3) / 51a₂+31
3a₂+2 / 3(51a₂+31)
4/3
4/3
3a₂+2 / 3(51a₂+31)
3(5a₂+3) / 51a₂+31
3(5a₂+3) / 51a₂+31
3(5a₂+3) / 51a₂+31
3(5a₂+3) / 51a₂+31
4/3
3a₂+2 / 3(51a₂+31)

$H_9$

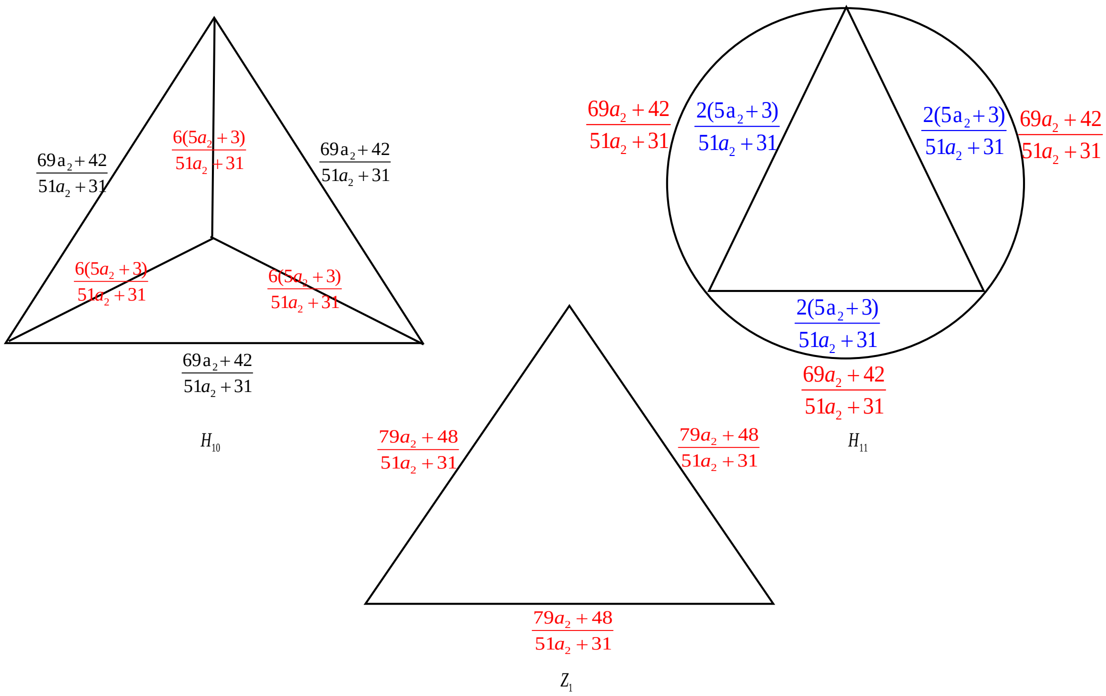



**Fig.(8):** The transformations from $Z_2$ to $Z_1$

Using the properties given in section 2 , we have the following the transformations:

$$\tau(H_1)=[\frac{1}{3}]^3\tau(Z_2), \tau(H_2)=\tau(H_1), \tau(H_3)=9a_2\tau(H_2), \tau(H_4)=(\frac{1}{3a_2+2})^3\tau(H_3),$$

$\tau(H_5)=\tau(H_4), \tau(H_6)=\frac{3a_2+2}{18a_2}\tau(H_5), \tau(H_7)=\tau(H_6), \tau(H_8)=9(\frac{5a_2+3}{3a_2+2})\tau(H_7), \tau(H_9)=[\frac{3(3a_2+2)}{51a_2+31}]^3\tau(H_8)$, $\tau(H_{10})=\tau(H_9), \tau(H_{11})=\frac{51a_2+31}{18(5a_2+3)}\tau(H_{10})$ and $\tau(Z_1)=\tau(H_{11})$. Combining these twelve transformations, we have

$$\tau(Z_2)=4(51a_2+31)^2\tau(Z_1). \qquad (6.1)$$

Further $\tau(Z_n)=\prod_{i=2}^{n}4(51a_i+31)^2\tau(Z_1)=3\times4^{n-1}a_1^2[\prod_{i=2}^{n}(51a_i+31)]^2$ (6.2)

where $a_{i-1}=\frac{79a_i+48}{51a_i+31}, i=2,3,\ldots,n$. Its characteristic equation is $17\lambda^2-16\lambda-16=0$ which have two roots

$\lambda_1=\frac{8-4\sqrt{21}}{17}$ and $\lambda_2=\frac{8+4\sqrt{21}}{17}$.

Subtracting these two roots into both sides of $a_{i-1}=\frac{79a_i+48}{51a_i+31}$, we get

$$a_{i-1}-\frac{8-4\sqrt{21}}{17}=\frac{79a_i+48}{51a_i+31}-\frac{8-4\sqrt{21}}{17}=(55+12\sqrt{21}).\frac{a_i-\frac{8-4\sqrt{21}}{17}}{51a_i+31} \qquad (6.3)$$

$$a_{i-1}-\frac{8+4\sqrt{21}}{17}=\frac{79a_i+48}{51a_i+31}-\frac{8+4\sqrt{21}}{17}=(55-12\sqrt{21}).\frac{a_i-\frac{8+4\sqrt{21}}{17}}{51a_i+31} \qquad (6.4)$$

Let $b_i=\frac{a_i-\frac{8-4\sqrt{21}}{17}}{a_i-\frac{8+4\sqrt{21}}{17}}$. Then by Eqs. (6.3) and (6.4), we get $b_{i-1}=(6049+1320\sqrt{21})b_i$ and $b_i=(6049+1320\sqrt{21})^{n-i}b_n$. Therefore.

$$a_i=\frac{(6049+1320\sqrt{21})^{n-i}(\frac{8+4\sqrt{21}}{17})b_n-\frac{8-4\sqrt{21}}{17}}{(6049+1320\sqrt{21})^{n-i}b_n-1}.$$

Thus

$$a_1=\frac{4(6049+1320\sqrt{21})^{n-1}(46+11\sqrt{21})+5(8-4\sqrt{21})}{(6049+1320\sqrt{21})^{n-1}(139+24\sqrt{21})+85}. \qquad (6.5)$$

Using the expression $a_{n-1}=\frac{79a_n+48}{51a_n+31}$ and denoting the coefficients of $79a_n+48$ and $51a_n+31$ as $\alpha_n$ and $\beta_n$, we have

$$51a_n+31=\alpha_0(79a_n+48)+\beta_0(51a_n+31),$$
$$51a_{n-1}+31=\frac{\alpha_1(79a_n+48)+\beta_1(51a_n+31)}{\alpha_0(79a_n+48)+\beta_0(51a_n+31)},$$
$$51a_{n-2}+31=\frac{\alpha_2(79a_n+48)+\beta_2(51a_n+31)}{\alpha_1(79a_n+48)+\beta_1(51a_n+31)},$$

⋮

$$51a_{n-i}+31=\frac{\alpha_i(79a_n+48)+\beta_i(51a_n+31)}{\alpha_{i-1}(79a_n+48)+\beta_{i-1}(51a_n+31)}, \quad (6.6)$$

$$51a_{n-(i+1)}+31=\frac{\alpha_{i+1}(79a_n+48)+\beta_{i+1}(51a_n+31)}{\alpha_i(79a_n+48)+\beta_i(51a_n+31)}, \quad (6.7)$$

⋮

$$51a_2+31=\frac{\alpha_{n-2}(79a_n+48)+\beta_{n-2}(51a_n+31)}{\alpha_{n-3}(79a_n+48)+\beta_{n-3}(51a_n+31)},$$

Substituting Eq.(6.6) into Eq.(6.2), we obtain

$$\tau(Z_n)=3\times 4^{n-1}a_1^2[\alpha_{n-2}(79a_n+48)+\beta_{n-2}(51a_n+31)]^2 \quad (6.8)$$

where $\alpha_0=0,\beta_0=1$ and $\alpha_1=51,\beta_1=31$. By the expression $a_{n-1}=\frac{79a_n+48}{51a_n+31}$ and Eqs. (6.6) and (6.7), we have

$$\alpha_{i+1}=110\alpha_i-\alpha_{i-1};\beta_{i+1}=110\beta_i-\beta_{i-1} \quad (6.9)$$

The characteristic equation of Eq. (6.9) is $\mu^2-110\mu+1=0$ which have two roots $\mu_1=55+12\sqrt{21}$ and $\mu_2=55-12\sqrt{21}$. The general solutions of Eq.(6.9) are $\alpha_i=c_1\mu_1^i+c_2\mu_2^i;\beta_i=d_1\mu_1^i+d_2\mu_2^i$.

Substituting the initial conditions $\alpha_0=0,\beta_0=1$ and $\alpha_1=51,\beta_1=31$, yields

$$\alpha_i=\frac{17\sqrt{21}}{168}(55+12\sqrt{21})^i-\frac{17\sqrt{21}}{168}(55-12\sqrt{21})^i;$$

$$\beta_i=(\frac{21-2\sqrt{21}}{42})(55+12\sqrt{21})^i+(\frac{21+2\sqrt{21}}{42})(55-12\sqrt{21})^i \quad (6.10)$$

If $a_n=1$, it means that $Z_n$ without any electrically equivalent transformation. Plugging Eq. (6.10) into Eq.(6.8), we have

$$\tau(Z_n)=3\times 4^{n-1}a_1^2[(\frac{2296+501\sqrt{21}}{56})(55+12\sqrt{21})^{n-2}+(\frac{2296-501\sqrt{21}}{56})(55-12\sqrt{21})^{n-2}]^2, n\geq 2. \quad (6.11)$$

When $n=1$ , $\tau(Z_1)=3$ which satisfies Eq.(6.11). Therefore the number of spanning trees in the sequence of $Z_n$ graph is given by

$$\tau(Z_n)=3\times 4^{n-1}a_1^2[(\frac{2296+501\sqrt{21}}{56})(55+12\sqrt{21})^{n-2}+(\frac{2296-501\sqrt{21}}{56})(55-12\sqrt{21})^{n-2}]^2, n\geq 1. \quad (6.12)$$

Where

$$a_1=\frac{4(6049+1320\sqrt{21})^{n-1}(46+11\sqrt{21})+5(8-4\sqrt{21})}{(6049+1320\sqrt{21})^{n-1}(139+24\sqrt{21})+85}. \ , n\geq 1. \quad (6.13)$$

Inserting Eq.(6.13) into Eq.(6.12) we obtain the result. □

## 7. Numerical Results

Table1: illustrates some the values of the number of spanning trees in the graphs $X_n$ , $T_n$ , $Y_n$ and $Z_n$.

| $n$ | $\tau(X_n)$ | $\tau(T_n)$ | $\tau(Y_n)$ | $\tau(Z_n)$ |
|---|---|---|---|---|
| 1 | 3 | 3 | 3 | 3 |
| 2 | 38988 | 47628 | 40344 | 193548 |
| 3 | 389606448 | 662815488 | 465904332 | 9366382128 |
| 4 | 3892947158208 | 9223739666112 | 5380263088224 | 453257961670848 |
| 5 | 38898321771725568 | 128357550588955392 | 62131276279240752 | 21934059131316880128 |
| 6 | 388671968855929801728 | 1786223526416070131712 | 717491956951619945856 | 1061432982230559089691648 |

## 8. Spanning Tree Entropy

After having explicit Formulas for the number of spanning trees of the sequence of the four families of graphs $X_n$, $T_n$, , $Y_n$ and $Z_n$, we can calculate its spanning tree entropy $Z$ which is a finite number and a very interesting quantity characterizing the network structure, defined as in [26,27] as: For a graph $G$ ,

$$Z(G)=\lim_{n\to\infty}\frac{\ln\tau(G)}{|V(G)|}. \quad (8.1)$$

$$Z(X_n)=\frac{1}{9}(\ln[4]+2\ln[25+4\sqrt{39}])=1.02328221,$$

$$Z(T_n)=\frac{2}{9}(\ln[59+\sqrt{3477}])=1.060088273,$$

$$Z(Y_n)=\frac{1}{9}(\ln[2]+2\ln[38+\sqrt{1443}])=1.039363057.$$

$$Z(Z_n)=\frac{1}{9}(\ln[4]+2\ln[55+12\sqrt{21}])=1.198565531.$$

Now we compare the value of entropy in our graphs with other graphs. It is clear that the entropy of the $Z_n$ graph is greater than the other three graphs and the entropy of the $X_n$ graph is smaller than the other three graphs. In addition the entropy of the our studied graphs $X_n, T_n$ and $Y_n$ is smaller than the fractal scale free lattice [28] which has the entropy 1.040 and two dimensional Sierpinski gasket [29] which has the entropy 1.166 of the same average degree 4, while the entropy of the $Z_n$ graph is larger than the entropy of the fractal scale free lattice and less than the entropy of two dimensional Sierpinski gasket.

## 9. Conclusions

In this work, we enumerate the number of spanning trees in the sequences of four sequences of graphs of average degree four based on Tridiminished icosahedron graph using electrically equivalent transformations. An advantage of this method lies in the avoidance of laborious computation of Laplacian spectra that is needed for a generic method for determining spanning trees.

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