# Spanning Trees Entropy of some of the families of graphs Based on a triangle

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**Abstract**

The number of spanning trees is an important quantity characterizing the reliability of a network. In this paper, we find the explicit formulas of the number of spanning trees of some new of the families of sequence graphs generated by triangle graph with its special feature in iteration. Using the electrically equivalent transformations, we obtain the weights of corresponding equivalent graphs and we further derive relationships for spanning trees between these graphs and transformed graphs. Finally, we compare the entropy of our graphs together and with other studied graphs of average degree.

**Keywords**: Number of spanning trees; Electrically equivalent transformations; Entropy; sequence graphs.

**Mathematics Subject Classification**: 97K30, 05C63.

## 1. Introduction

Calculating the spanning trees number in a graph is one of the main studied problems in graph theory.

A spanning tree of a connected graph $G$ with n vertices is a connected $(n-1)-$ edge subgraph of $G$. The number of spanning trees of a graph $G$ denoted by$\tau(G)$, also called the complexity of $G$ [1], is an important, well-studied quantity in graph theory, and appears in a number of applications. Most memorable application fields are network reliability [2], recounting certain chemical isomers [3], and counting the number of Eulerian circuits in a graph [1]. In particular, counting spanning trees is an essential step in many methods for computing, bounding, and approximating network reliability [4]. In a network modeled by a graph, intercommunication between all nodes of the network implies that the graph must contain a spanning tree and, thus, maximizing the number of spanning trees is a way of maximizing reliability.

In 1847, a classical result of Kirchhoff [5] can be used to determine the number of spanning trees for a connected graph $G=(V,E)$ with $n$ vertices$\{v_1,v_2,\ldots,v_n\}$, and the Kirchhoff matrix $L$ defined as $n\times n$ characteristic matrix$L=D-A$, where $D$ is the diagonal matrix of the degrees of $G$ and $A$ is the adjacency matrix of $G$, $L=[a_{ij}]$ defined as follows:

$$L=[a_{ij}]=\begin{cases} deg(v_i) & \text{if } i=j \\ -1 & \text{if } (v_i,v_j)\in E(G) \\ 0 & \text{if } (v_i,v_j)\notin E(G) \end{cases}$$

All of co-factors of $L$ are equal to the number of spanning trees of the graph $G$ .

Another method for calculating $\tau(G)$ . Let $\mu_1\geq\mu_2\geq\ldots\geq\mu_n=0$ denote the eigenvalues of $L$ matrix of a graph $G$ with $n$ vertices. . In 1974, Kelmans and Chelnokov [6] has shown that

$$\tau(G)=\frac{1}{n}\prod_{i=1}^{n-1}\mu_i. \tag{1.1}$$

One common method for finding the number of spanning tress, $\tau(G)$, is the deletion-contraction method. This method is a dependable method which allows to calculate the number of spanning trees of a multigraph $G$ . This method uses the fact that

$$\tau(G)=\tau(G-e)+\tau(G/e) \tag{1.2}$$

where $G-e$ denotes the graph obtained by deleting an arbitrary edge $e$, and $G/e$ denotes the graph obtained by contracting an arbitrary edge $e$ [1,7]. For more methods and other techniques see [8-15]

## 2. Electrically Equivalent Transformations

An edge-weighted graph, whose weights represent the conductance of the corresponding edges, may be thought of as an electrical network, which is why Kirchhoff was motivated to research electrical networks. The quotient of the (weighted) number of spanning trees and the (weighted) number of so-called thickets—that is, spanning forests with exactly two components and the characteristic that each component contains precisely one of the vertices $u,v$ can be used to express the effect conductance between two vertices $u,v$ [16,117,18,19]. The impact of a few basic modifications on the quantity of spanning trees is listed below. The weighted number of spanning trees $G$ is indicated by $\tau(G)$ and let $G$ be an edge weighted graph and $G'$ be the associated electrically equivalent graph.

• **Parallel edges:** When two parallel edges in $G$, each with conductances $u$ and $v$, are merged into a single edge in $G'$ with a conductance of $u+v$, the count of spanning trees, $\tau(G')$, remains unchanged compared to $\tau(G)$.

• **Serial edges:** If two serial edges in $G$, with conductances $u$ and $v$, are combined into a single edge in $G'$ with a conductance of $uv/(u+v)$, then $\tau(G')$ can be calculated as $(1/(u+v)$ multiplied by $\tau(G)$.

• **Δ-Y Transformation:** When a triangle in $G$, with conductances $x,y$ and $z$ is transformed into an electrically equivalent star graph in $G'$ with conductances $x=(uv+vw+wu/u, y=(uv+vw+wu/v,$ and $z=(uv+vw+wu)/w$, the count of spanning trees in $G',\tau(G')$, can be determined as $(uv+vw+wu)^2/uvw$ multiplied by $\tau(G)$.

• **Y-Δ Transformation:** If a star graph in $G$, with conductances $u, v$ and $w$, is converted into an electrically equivalent triangle in $G'$ with conductances $x = vw/(u+v+w), y = uw/(u+v+w)$ and $z = uv/(u+v+w)$, then $\tau(G')$ is given by $1/(u+v+w)$ multiplied by $\tau(G)$.

In mathematics, it is common to derive new structures from existing ones. This principle extends to graphs, where numerous new graphs can be generated from a given set. In this study, we determine the complexity for four novel types of graphs of the same average degree we named it $\Gamma_1^{(n)}, \Gamma_2^{(n)}$ *and* $\Gamma_3^{(n)}$ respectively.

## 3. Number of spanning trees in the sequences of $\Gamma_1^{(n)}$ graph

The graph $\Gamma_1^{(n)}$ is defined recursively using the graphs $\Gamma_1^{(1)}$ (triangle or $K_3$) and $\Gamma_1^{(2)}$ as shown in Figure 1. The graph $\Gamma_1^{(n)}$, $n = 3$ is obtained by replacing the central triangle in $\Gamma_1^{(2)}$ by a copy of $\Gamma_1^{(2)}$. In general, $\Gamma_1^{(n)}$ is obtained by replacing the central triangle in $\Gamma_1^{(n-1)}$ with $\Gamma_1^{(2)}$. According to this construction, the number of total vertices $|V(\Gamma_1^{(n)})|$ and edges $|E(\Gamma_1^{(n)})|$ are $|V(\Gamma_1^{(n)})| = 9n - 6$ and $|E(\Gamma_1^{(n)})| = 24n - 21$, $n = 1, 2, \ldots$. The average degree of $\Gamma_1^{(n)}$ is in the large $n$ limit which is $\frac{16}{3}$.

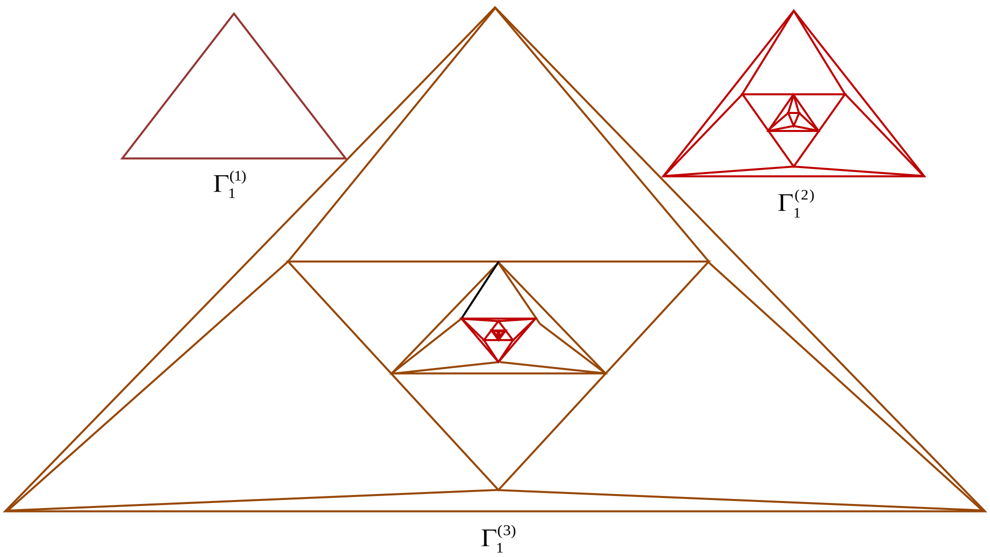
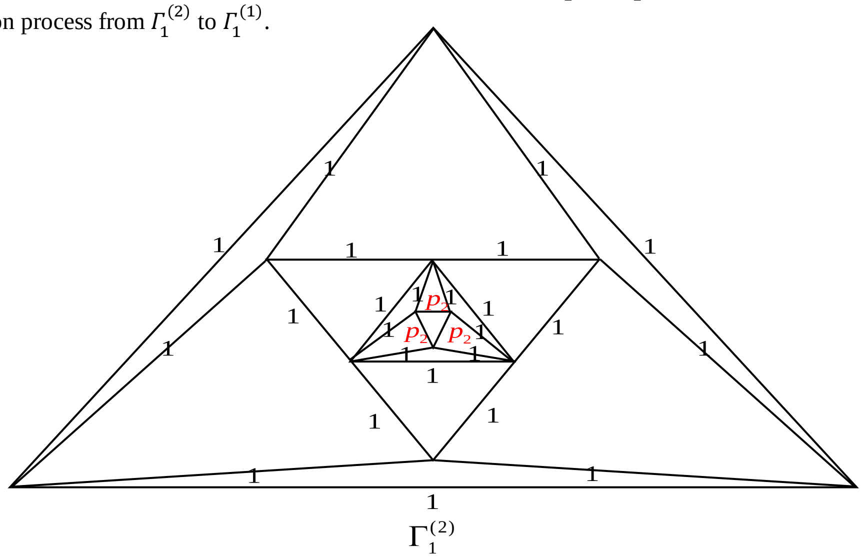
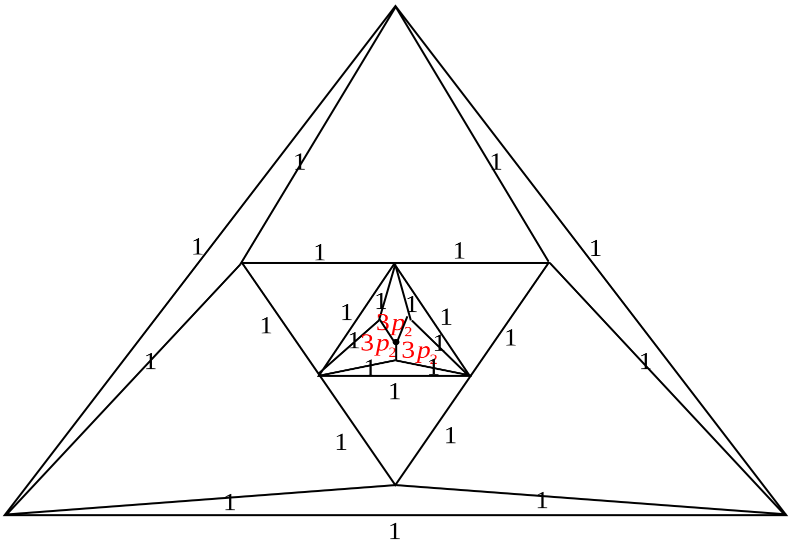



**Fig.1** Some sequences of the graph $\Gamma_1^{(n)}$

**Theorem 1.** For $n \geq 1$, the number of spanning trees in the sequence of the graph $\Gamma_1^{(n)}$ is given by
$(32^{3n-7}((5170 - 697\sqrt{55})(89 + 12\sqrt{55})^n + (89 - 12\sqrt{55})^n(5170 + 697\sqrt{55}))^2(247 - 76\sqrt{55} + (1073 + 160\sqrt{55})(15841 + 2136\sqrt{55})^{-1+n})^2)/\ (3025(513 + (807 + 84\sqrt{55})(15841 + 2136\sqrt{55})^{n-1})^2)$

Proof: We use the electrically equivalent transformation to transform $\Gamma_1^{(i)}$ to $\Gamma_1^{(i-1)}$. Fig.2 illustrates the transformation process from $\Gamma_1^{(2)}$ to $\Gamma_1^{(1)}$.

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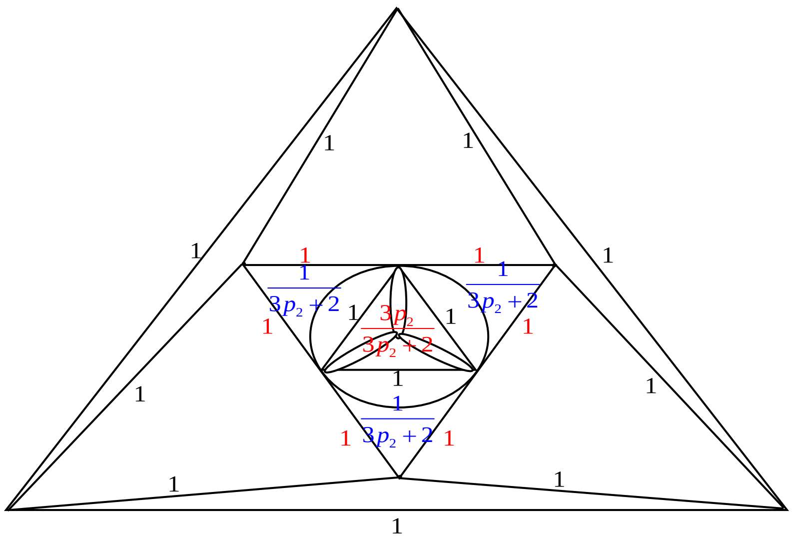
$3p_2$
$3p_2$
$3p_2$

$G_1$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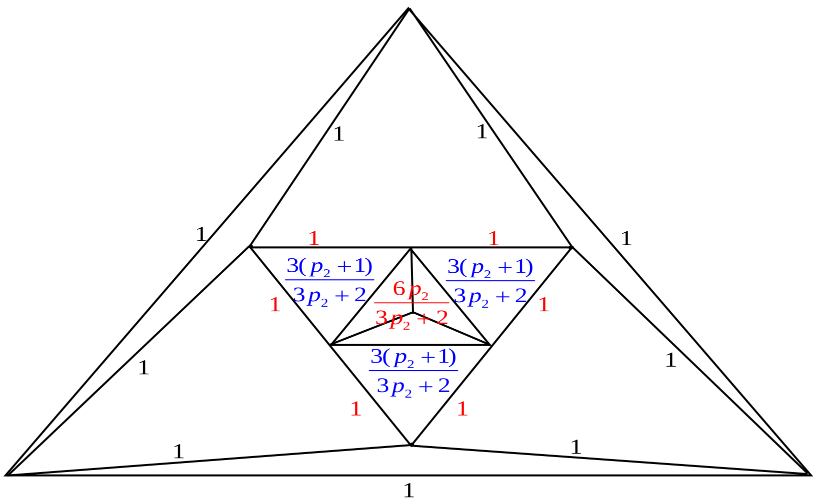
$\frac{1}{3p_2+2}$

$G_2$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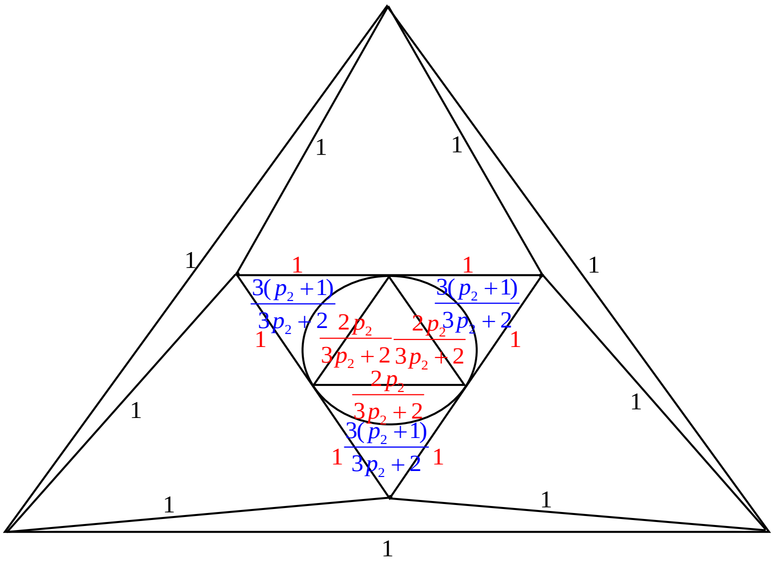
$\frac{3(p_2+1)}{3p_2+2}$

$G_3$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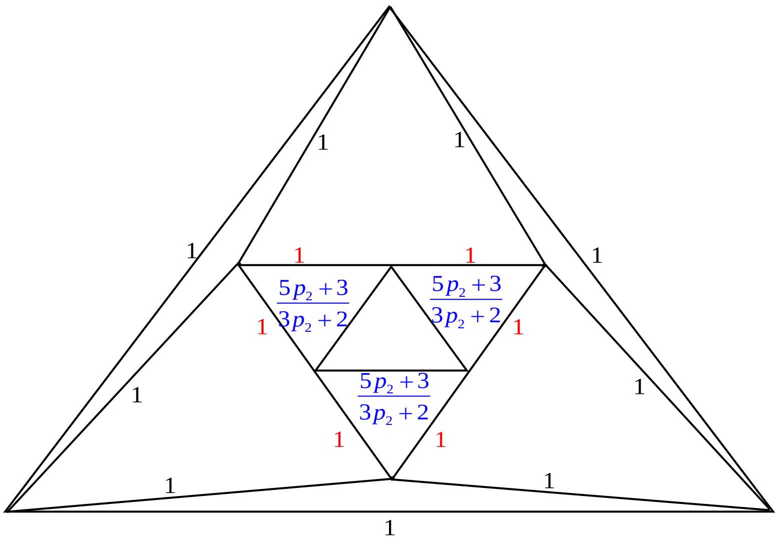
$G_4$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5p2 +3 / 3p2 +2
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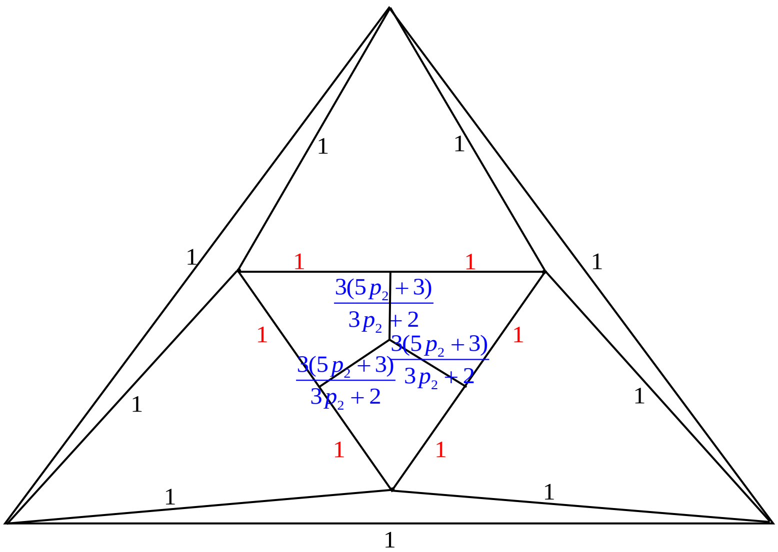
$G_5$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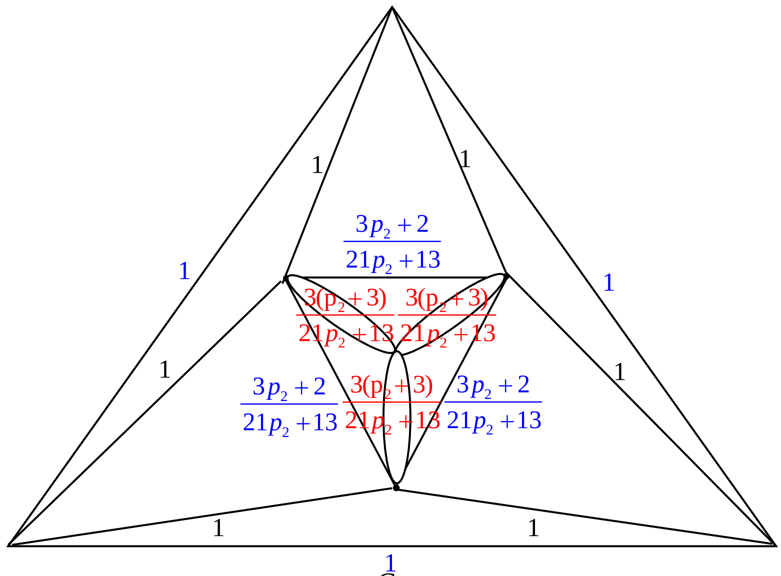
$G_6$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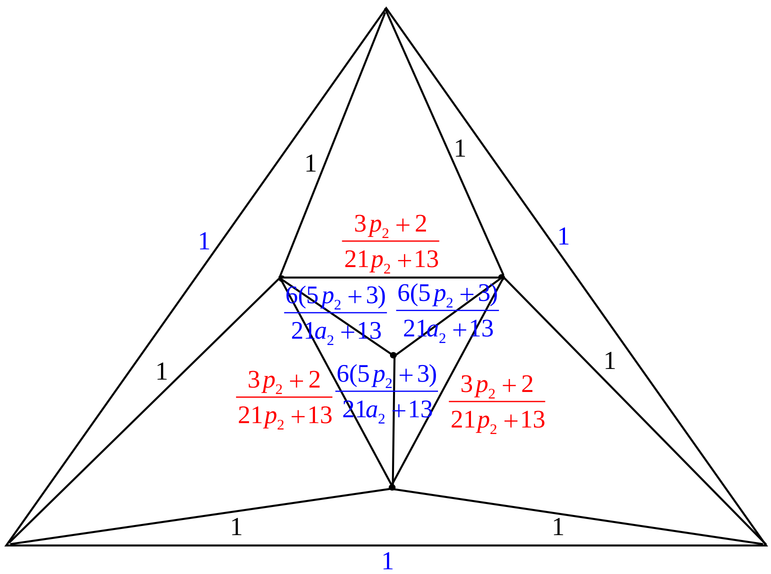
$G_7$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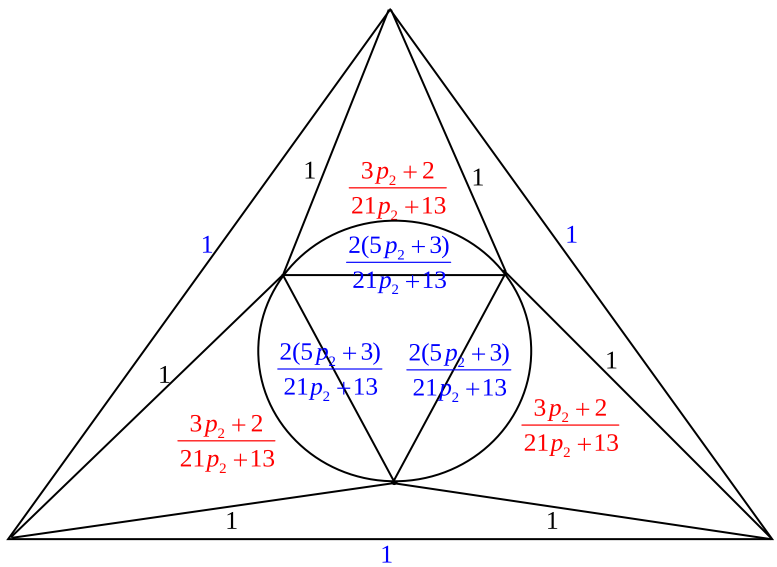
$G_8$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$G_9$

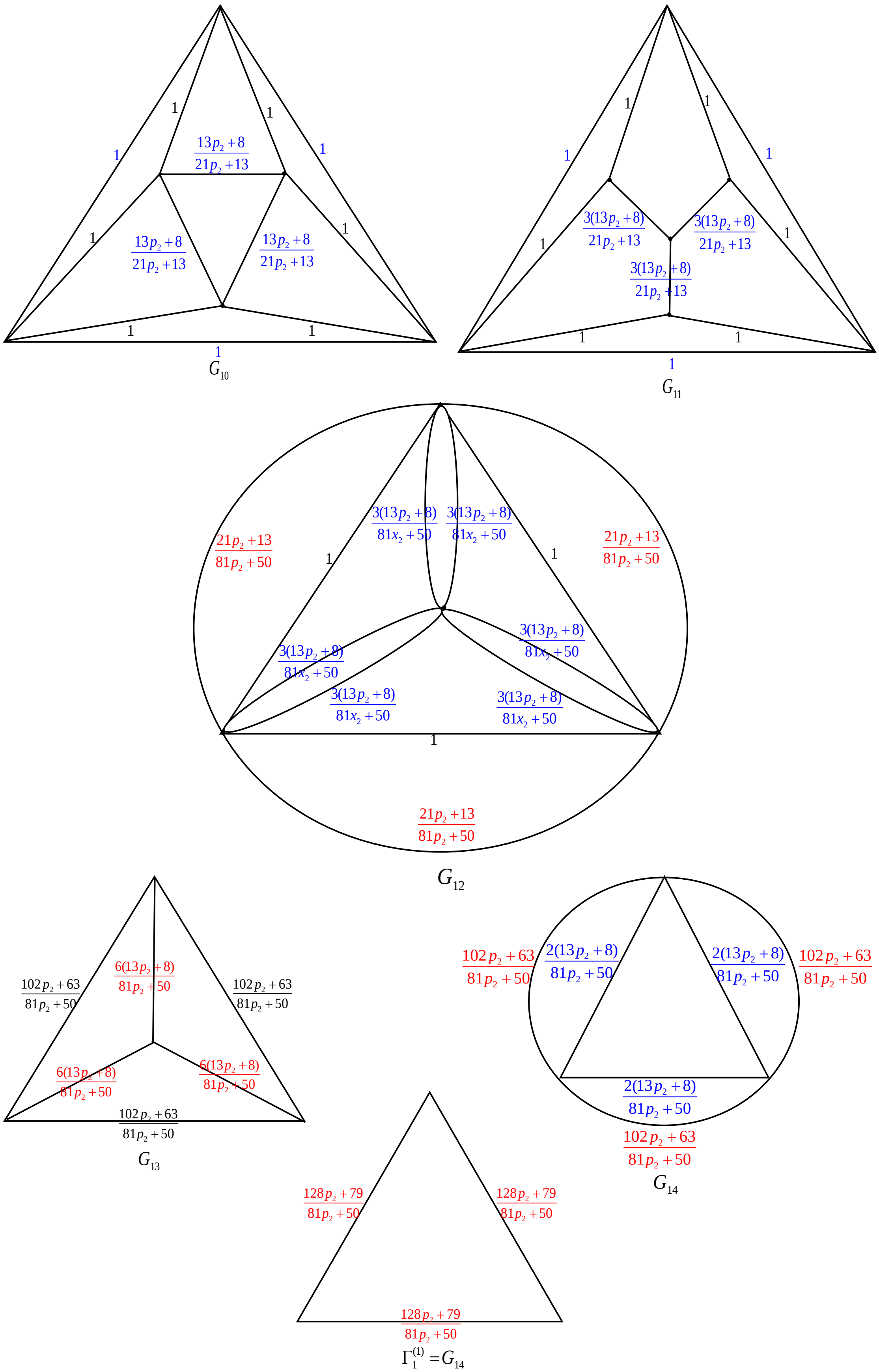



**Fig.2** The transformations from $\Gamma_1^{(2)}$ to $\Gamma_1^{(1)}$

Using the properties given in section 2, we have the following the transformations:
$\tau(G_1)=9p_2\tau(\Gamma_1^{(2)}),\tau(G_2)=[\frac{1}{3p_2+2}]^3\tau(G_1),\tau(G_3)=\tau(G_2),\tau(G_4)=[\frac{3p_2+2}{18p_2}]\tau(G_3),\tau(G_5)=\tau(G_4),$
$\tau(G_6)=9[\frac{5p_2+3}{3p_2+2}]\tau(G_5),\tau(G_7)=[\frac{3p_2+2}{21p_2+13}]^3\tau(G_6),\tau(G_8)=\tau(G_7),\tau(G_9)=[\frac{21p_2+13}{18(5p_2+3)}]\tau(G_8)\ \tau(G_{10})=$
$\tau(G_9),\tau(G_{11})=9[\frac{13p_2+8}{21p_2+13}]\tau(G_{10}),\tau(G_{12})=[\frac{21p_2+13}{81p_2+50}]^3\tau(G_{11}),\tau(G_{13})=\tau(G_{12}),\tau(G_{14})=[\frac{81p_2+50}{18(13p_2+8)}]\tau(G_{13})$
and $\tau(\Gamma_1^1)=\tau(G_{14})$.
Combining these fifteen transformations, we have:

$$\tau(\Gamma_1^{(2)})=8(81p_2+50)^2\tau(\Gamma_1^{(1)}). \tag{3.1}$$

Further

$$\tau(\Gamma_1^{(n)})=\prod_{i=2}^{n}8(81p_i+50)^2\tau(\Gamma_1^{(1)})=3\times 8^{n-1}p_1^2[\prod_{i=2}^{n}(81p_i+50)]^2, \tag{3.2}$$

where $r_{i-1}=\frac{128p_i+79}{81p_i+50}, i=2,3,\dots,n.$

Its characteristic equation is $81\theta^2-78\theta-79=0$ with roots $\theta_1=\frac{13-4\sqrt{55}}{27}$ and $\theta_2=\frac{13+4\sqrt{55}}{27}$.
Subtracting these two roots into both sides of $r_{i-1}=\frac{128p_i+79}{81p_i+50}$, we get

$$r_{i-1}-\frac{13-4\sqrt{55}}{27}=\frac{128p_i+79}{81p_i+50}-\frac{13-4\sqrt{55}}{27}=\frac{(89+12\sqrt{55})(p_i-\frac{13-4\sqrt{55}}{27})}{(81p_i+50)} \tag{3.3}$$

$$r_{i-1}-\frac{13+4\sqrt{55}}{27}=\frac{128p_i+79}{81p_i+50}-\frac{13+4\sqrt{55}}{27}=\frac{(89-12\sqrt{55})(p_i-\frac{13+4\sqrt{55}}{27})}{(81p_i+50)} \tag{3.4}$$

Let $q_i=\frac{p_i-\frac{13-4\sqrt{55}}{27}}{p_i-\frac{13+4\sqrt{55}}{27}}$.Then by Eqs. (3.3) and (3.4), we get $q_{i-1}=(15841+2136\sqrt{55})q_i$ and $q_i=(15841+2136\sqrt{55})^{n-i}q_n$.

Therefore $r_i=\frac{(15841+2136\sqrt{55})^{n-i}(\frac{13+4\sqrt{55}}{27})q_n-(\frac{13-4\sqrt{55}}{27})}{(15841+2136\sqrt{55})^{n-i}q_n-1}$.

Thus

$$r_1=\frac{(15841+2136\sqrt{55})^{n-1}(\frac{13+4\sqrt{55}}{27})q_n-(\frac{13-4\sqrt{55}}{27})}{(15841+2136\sqrt{55})^{n-1}q_n-1}. \tag{3.5}$$

Using the expression $r_{n-1}=\frac{128p_n+79}{81p_n+50}$ and denoting the coefficients of $128r_n+79$ and $81p_n+50$ as $a_n$ and $b_n$ , we have

$$81r_n+50=a_0(128r_n+79)+b_0(81r_n+50),$$
$$81r_{n-1}+50=\frac{a_1(128r_n+79)+b_1(81r_n+50)}{a_0(128r_n+79)+b_0(81r_n+50)},$$
$$81r_{n-2}+50=\frac{a_2(128r_n+79)+b_2(81r_n+50)}{a_1(128r_n+79)+b_1(81r_n+50)},$$
$$\vdots$$

$$81r_{n-i}+50=\frac{a_i(128r_n+79)+b_i(81r_n+50)}{a_{i-1}(128r_n+79)+b_{i-1}(81r_n+50)}, \tag{3.6}$$

$$81r_{n-(i+1)}+50=\frac{a_{i+1}(128r_n+79)+b_{i+1}(81r_n+50)}{a_i(128r_n+79)+b_i(81r_n+50)}, \tag{3.7}$$

$$\vdots$$

$$81r_2+50=\frac{a_{n-2}(128r_n+79)+b_{n-2}(81r_n+50)}{a_{n-3}(128r_n+79)+b_{n-3}(81r_n+50)},$$

Thus, we obtain

$$\tau(\Gamma_1^{(2)})=3\times 8^{n-1}r_1^2[a_{n-2}(128r_n+79)+b_{n-2}(81r_n+50)]^2 \tag{3.8}$$

where $a_0=0,b_0=1$ and $a_1=81,b_1=50$. By the expression $r_{n-1}=\frac{128r_n+79}{81r_n+50}$ and using Eqs. (3.6) and (3.7), we have

$$a_{i+1}=178a_i-a_{i-1};\ b_{i+1}=178b_i-b_{i-1} \tag{3.9}$$

The characteristic equation of Eq. (3.9) is $\varphi^2-178\varphi+1=0$ with roots $\varphi_1=89+12\sqrt{55}$ and $\varphi_2=89+12\sqrt{55}$. The general solutions of Eq. (3.9) are
$a_i=h_1\varphi_1^i+h_2\varphi_2^i;\ b_i=k_1\varphi_1^i+k_2\varphi_2^i.$
Using the initial conditions $a_0=0,b_0=1$ and $a_1=81,b_1=50$, yields

$$a_i=\frac{27\sqrt{55}}{440}(89+12\sqrt{55})^i-\frac{27\sqrt{55}}{440}(89-12\sqrt{55})^i;$$

$$b_i=(\frac{220-13\sqrt{55}}{440})(89+12\sqrt{55})^i+(\frac{220+13\sqrt{55}}{440})(89-12\sqrt{55})^i \tag{3.10}$$

If $r_n=1$, it means that $\Gamma_1^{(n)}$ is without any electrically equivalent transformation. Plugging Eq. (3.10) into Eq. (3.8), we have

$$\tau(\Gamma_1^{(1)})=3\times 8^{n-1}r_1^2[(\frac{28820+3886\sqrt{55}}{440})(89+12\sqrt{55})^{n-2}+(\frac{28820-3886\sqrt{55}}{440})(89-12\sqrt{55})^{n-2}]^2, n\geq 2. \tag{3.11}$$

When $n=1$ , $\tau(\Gamma_1^{(1)})=3$ which satisfies Eq. (3.11). Therefore, the number of spanning trees in the sequence of the graph $\Gamma_1^{(n)}$ is given by

$$\tau(\Gamma_1^{(1)}) = 3\times 8^{n-1} r_1^2 [(\tfrac{28820+3886\sqrt{55}}{440})(89+12\sqrt{55})^{n-2} + (\tfrac{28820-3886\sqrt{55}}{440})(89-12\sqrt{55})^{n-2}]^2, n\geq 1. \quad (3.12)$$

where

$$r_1 = \frac{(15841+2136\sqrt{55})^{n-1}(1073+160\sqrt{55})+19(13-4\sqrt{55})}{3(15841+2136\sqrt{55})^{n-1}(269+28\sqrt{55})+513}, n\geq 1. \quad (3.13)$$

Inserting Eq. (3.13) into Eq.(3.12) we obtain the desired result. □

## 4. Number of spanning trees in the sequences of $\Gamma_2^{(n)}$ graph

The graph $\Gamma_2^{(n)}$ is defined recursively using the graphs $\Gamma_2^{(1)}$ (triangle or $K_3$) and $\Gamma_2^{(2)}$ as shown in Figure 3. The graph $\Gamma_2^{(n)}$, $n = 3$ is obtained by replacing the central triangle in $\Gamma_2^{(2)}$ by a copy of $\Gamma_2^{(2)}$. In general, $\Gamma_2^{(n)}$ is obtained by replacing the central triangle in $\Gamma_2^{(n-1)}$ with $\Gamma_2^{(2)}$. According to this construction, the number of total vertices $|V(\Gamma_2^{(n)})|$ and edges $|E(\Gamma_2^{(n)})|$ are $|V(\Gamma_2^{(n)})| = 9n-6$ and $|E(\Gamma_2^{(n)})| = 24n-21$, $n = 1, 2, \ldots$. The average degree of $\Gamma_2^{(n)}$ is in the large $n$ limit which is $\frac{16}{3}$.

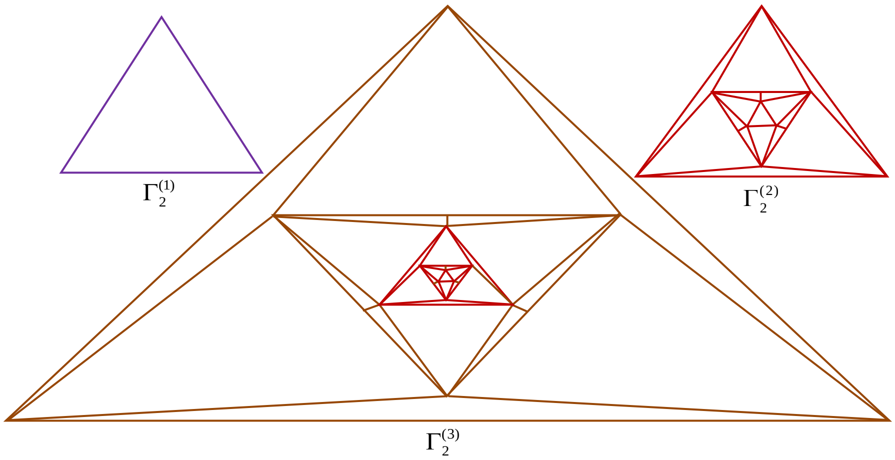
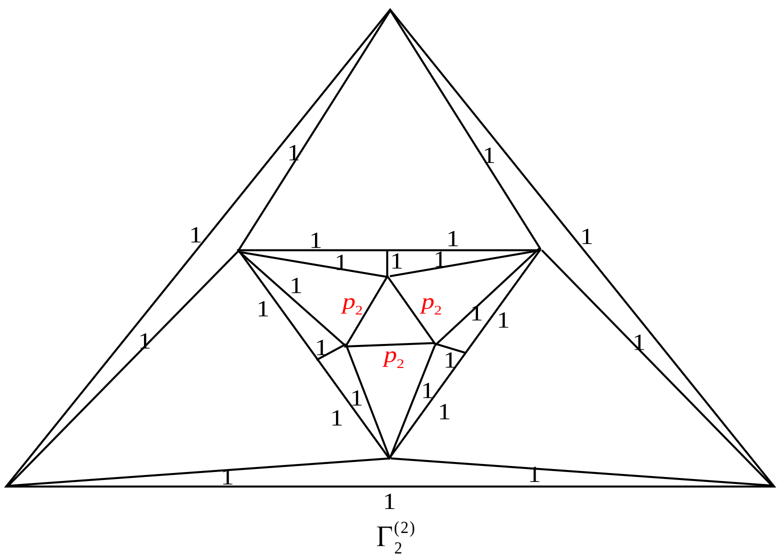



**Fig. 3** Some sequences of the graph $\Gamma_2^{(n)}$

**Theorem 2.** For $n\geq 1$, the number of spanning trees in the sequence of the graph $\Gamma_2^{(n)}$ is given by
$(34^{2n-7}((7657-125\sqrt{3705})(61+\sqrt{3705})^n + (61-\sqrt{3705})^n(7657+125\sqrt{3705}))^2(-11(-21+\sqrt{3705}) + 8^{1-n}(3713+61\sqrt{3705})^n(-8643+142\sqrt{3705}))^2)/\,(61009(561+3(\frac{1}{8}(3713-61\sqrt{3705}))^{1-n}(307+4\sqrt{3705}))^2)$

Proof: We use the electrically equivalent transformation to transform $\Gamma_2^{(i)}$ to $\Gamma_2^{(i-1)}$. Fig.4 illustrates the transformation process from $\Gamma_2^{(2)}$ to $\Gamma_2^{(1)}$.



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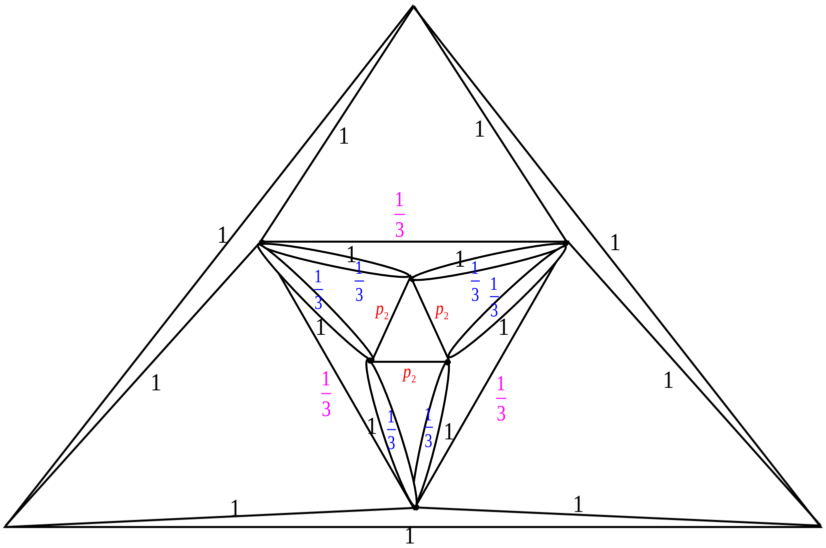
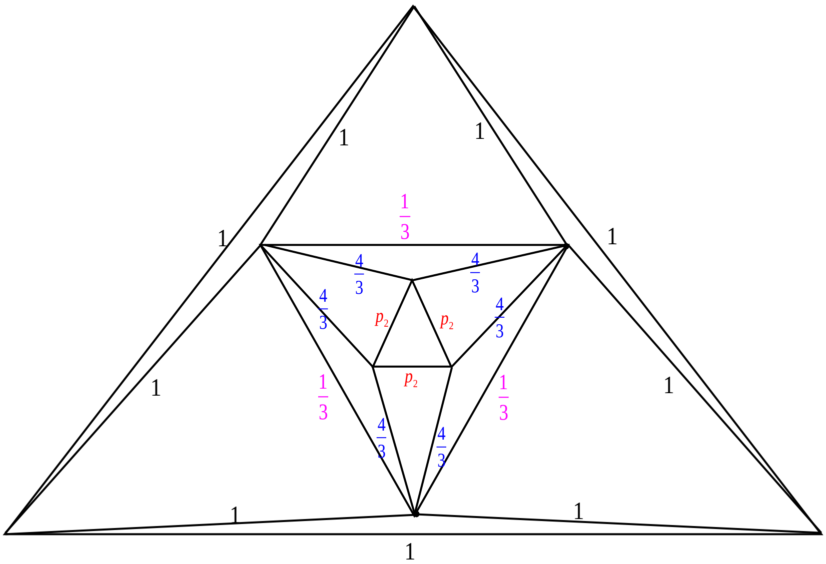
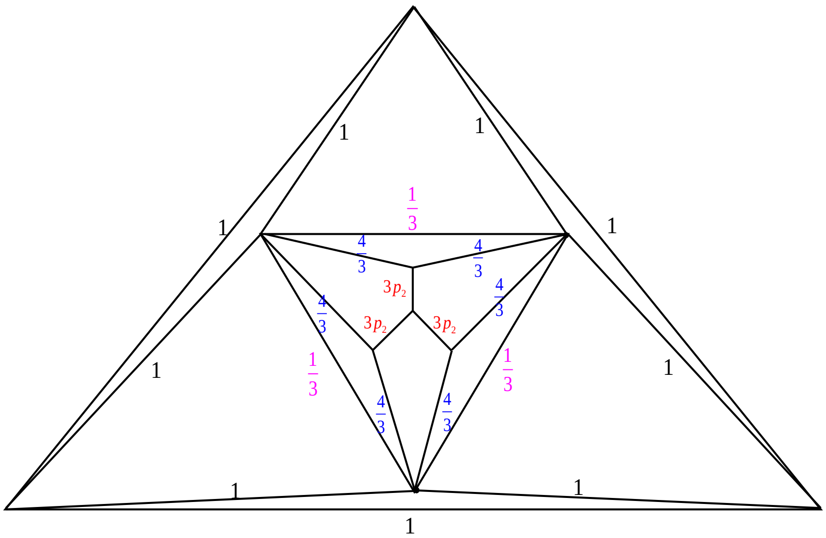


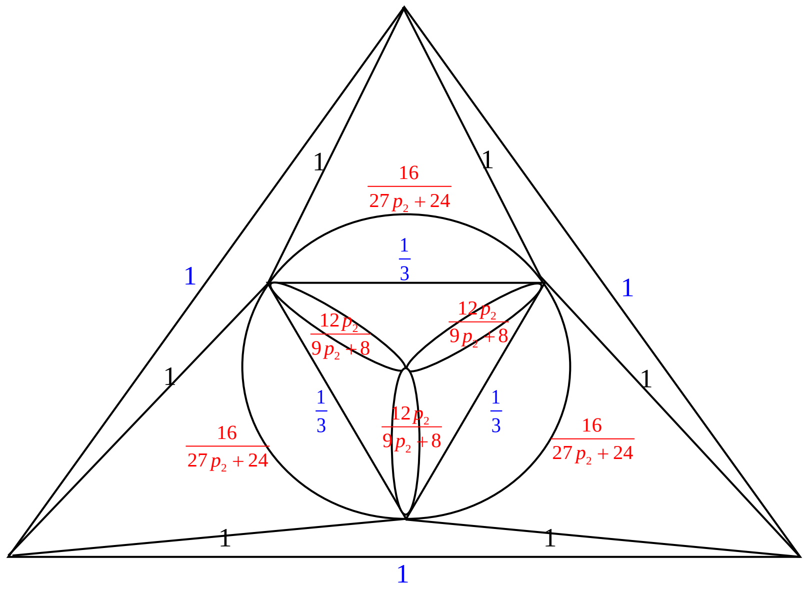
$G_1$



$G_2$



$G_3$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$\frac{1}{3}$
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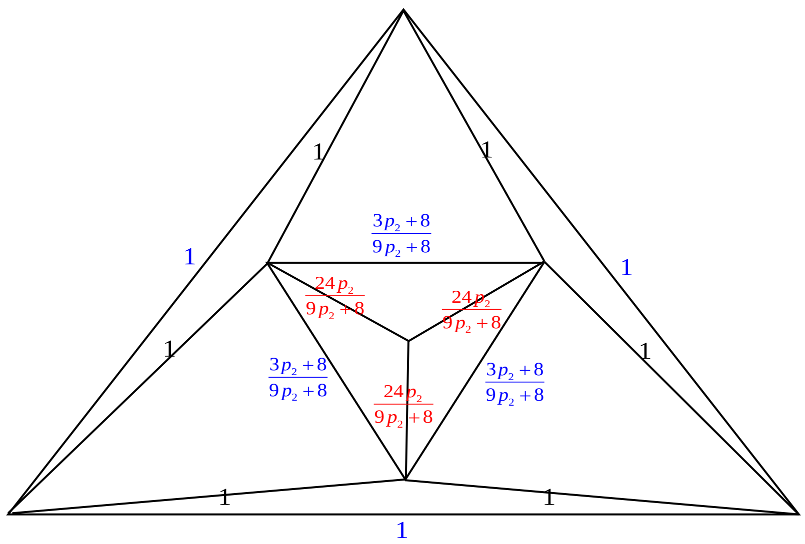
$\frac{12p_2}{9p_2+8}$

$G_4$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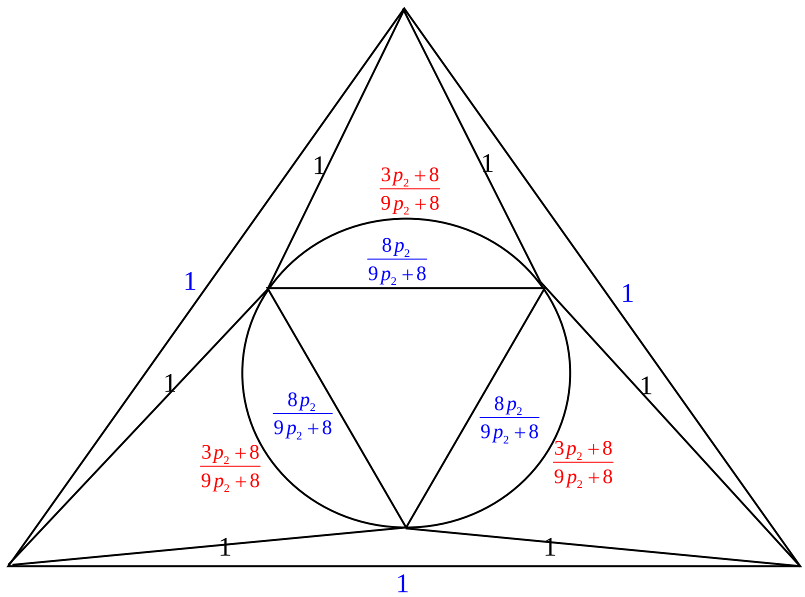
$\frac{24p_2}{9p_2+8}$
$\frac{24p_2}{9p_2+8}$
$\frac{24p_2}{9p_2+8}$

$G_5$


1
1
1
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1
$\frac{3p_2+8}{9p_2+8}$
$\frac{3p_2+8}{9p_2+8}$
$\frac{3p_2+8}{9p_2+8}$
$\frac{8p_2}{9p_2+8}$
$\frac{8p_2}{9p_2+8}$
$\frac{8p_2}{9p_2+8}$

$G_6$

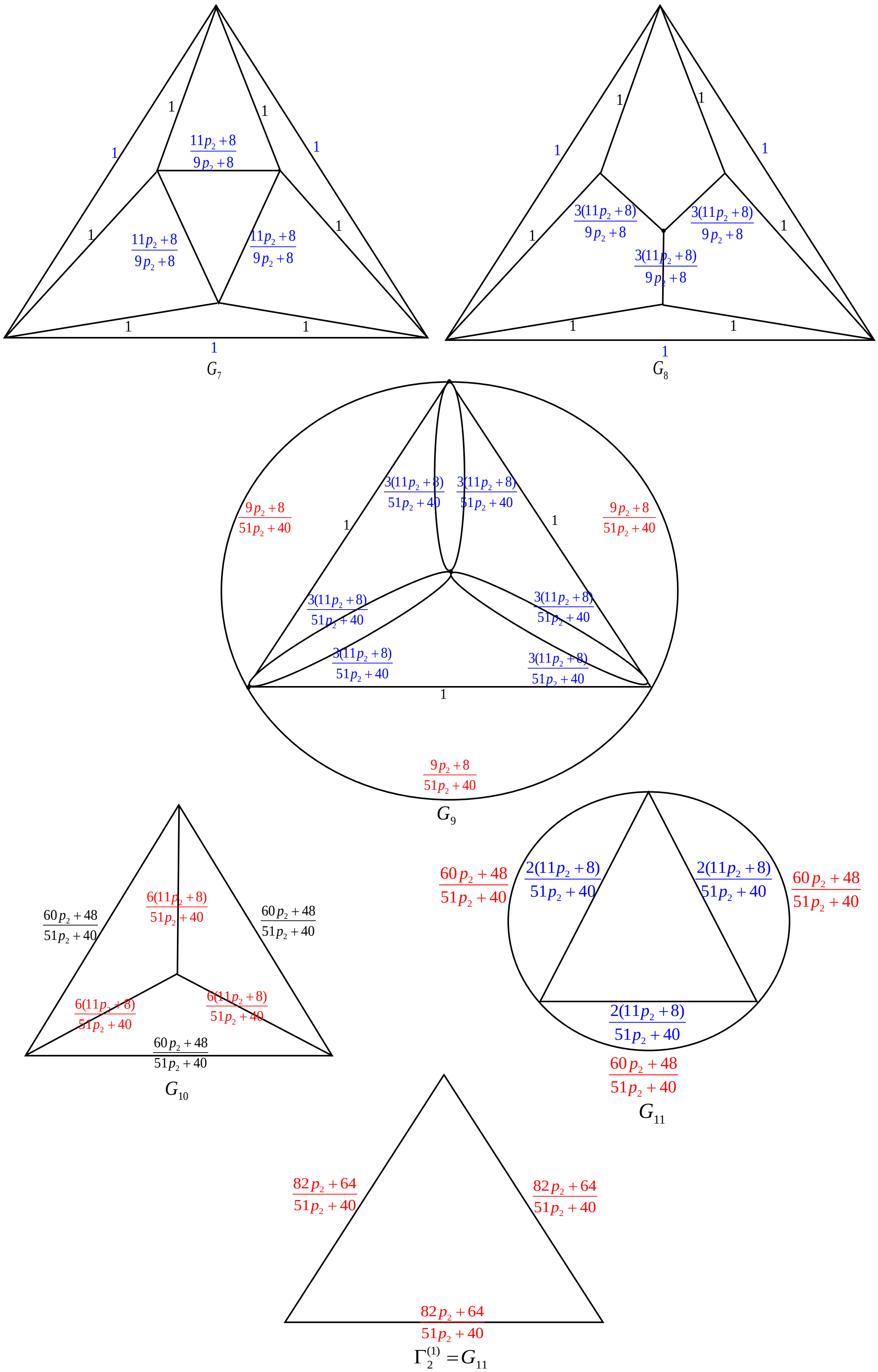



**Fig.4** The transformations from $\Gamma_2^{(2)}$ to $\Gamma_2^{(1)}$

Using the properties given in section 2, we have the following the transformations:
$\tau(G_1)=[\frac{1}{3}]^3\tau(\Gamma_2^{(2)}),\tau(G_2)=\tau(G_1),\tau(G_3)=9p_2\tau(G_2),\tau(G_4)=[\frac{3}{9p_2+8}]^3\tau(G_3),\tau(G_5)=\tau(G_4),$
$\tau(G_6)=[\frac{9p_2+8}{72p_2}]\tau(G_5),\tau(G_7)=\tau(G_6),\tau(G_8)=9[\frac{11p_2+8}{9p_2+8}]\tau(G_7),\tau(G_9)=[\frac{9p_2+8}{51p_2+40}]^3\tau(G_8),\tau(G_{10})=$
$\tau(G_9),\tau(G_{11})=[\frac{51p_2+40}{18(11p_2+8)}]\tau(G_{10})$, and $\tau(\Gamma_2^1)=\tau(G_{11})$.
Combining these twelve transformations, we have

$$\tau(\Gamma_2^{(2)})=16(51p_2+40)^2\tau(\Gamma_2^{(1)}). \tag{4.1}$$

Further

$$\tau(\Gamma_2^{(n)})=\prod_{i=2}^{n}16(51p_i+40)^2\tau(\Gamma_2^{(1)})=3\times16^{n-1}p_1^2[\prod_{i=2}^{n}(51p_i+40)]^2, \tag{4.2}$$

where $r_{i-1}=\frac{82p_i+64}{51p_i+40}, i=2,3,\ldots,n$.

Its characteristic equation is $51\theta^2-42\theta-64=0$ with roots $\theta_1=\frac{21-\sqrt{3705}}{51}$ and $\theta_2=\frac{21+\sqrt{3705}}{51}$. Subtracting these two roots into both sides of $r_{i-1}=\frac{82p_i+64}{51p_i+40}$, we get

$$r_{i-1}-\frac{21-\sqrt{3705}}{51}=\frac{82p_i+64}{51p_i+40}-\frac{21-\sqrt{3705}}{51}=\frac{(61+\sqrt{3705})(p_i-\frac{21-\sqrt{3705}}{51})}{(51p_i+40)} \tag{4.3}$$

$$r_{i-1}-\frac{21+\sqrt{3705}}{51}=\frac{82p_i+64}{51p_i+40}-\frac{21+\sqrt{3705}}{51}=\frac{(61-\sqrt{3705})(p_i-\frac{21+\sqrt{3705}}{51})}{(51p_i+40)} \tag{4.4}$$

Let $q_i=\frac{p_i-\frac{21-\sqrt{3705}}{51}}{p_i-\frac{21+\sqrt{3705}}{51}}$.Then by Eqs. (4.3) and (4.4), we get $q_{i-1}=(\frac{3713+61\sqrt{3705}}{8})q_i$ and $q_i=(\frac{3713+61\sqrt{3705}}{8})^{n-i}q_n$

Therefore $r_i=\frac{(\frac{3713+61\sqrt{3705}}{8})^{n-i}(\frac{21+\sqrt{3705}}{51})q_n-(\frac{21-\sqrt{3705}}{51})}{(\frac{3713+61\sqrt{3705}}{8})^{n-i}q_n-1}$.

Thus

$$r_1=\frac{(\frac{3713+61\sqrt{3705}}{8})^{n-1}(\frac{21+\sqrt{3705}}{51})q_n-(\frac{21-\sqrt{3705}}{51})}{(\frac{3713+61\sqrt{3705}}{8})^{n-1}q_n-1}. \tag{4.5}$$

Using the expression $r_{n-1}=\frac{82p_n+64}{51p_n+40}$ and denoting the coefficients of $82r_n+64$ and $51p_n+40$ as $a_n$ and $b_n$ , we have

$$51r_n+40=a_0(82r_n+64)+b_0(51r_n+40),$$
$$51r_{n-1}+40=\frac{a_1(82r_n+64)+b_1(51r_n+40)}{a_0(82r_n+64)+b_0(51r_n+40)},$$
$$51r_{n-2}+40=\frac{a_2(82r_n+64)+b_2(51r_n+40)}{a_1(82r_n+64)+b_1(51r_n+40)},$$
$$\vdots$$

$$51r_{n-i}+40=\frac{a_i(82r_n+64)+b_i(51r_n+40)}{a_{i-1}(82r_n+64)+b_{i-1}(51r_n+40)}, \tag{4.6}$$

$$51r_{n-(i+1)}+40=\frac{a_{i+1}(82r_n+64)+b_{i+1}(51r_n+40)}{a_i(82r_n+64)+b_i(51r_n+40)}, \tag{4.7}$$

$$\vdots$$

$$51r_2+40=\frac{a_{n-2}(82r_n+64)+b_{n-2}(51r_n+40)}{a_{n-3}(82r_n+64)+b_{n-3}(51r_n+40)},$$

Thus, we obtain

$$\tau(\Gamma_2^{(2)})=3\times16^{n-1}r_1^2[a_{n-2}(82r_n+64)+b_{n-2}(51r_n+40)]^2 \tag{4.8}$$

where $a_0=0, b_0=1$ and $a_1=51, b_1=40$.

By the expression $r_{n-1}=\frac{82r_n+64}{51r_n+40}$ and using Eqs. (4.6) and (4.7), we have

$$a_{i+1}=122a_i-16a_{i-1}; b_{i+1}=122b_i-16b_{i-1} \tag{4.9}$$

The characteristic equation of Eq. (4.9) is $\varphi^2-122\varphi+16=0$ with roots $\varphi_1=61+\sqrt{3705}$ and $\varphi_2=61-\sqrt{3705}$. The general solutions of Eq. (4.9) are $a_i=h_1\varphi_1^i+h_2\varphi_2^i; b_i=k_1\varphi_1^i+k_2\varphi_2^i$.

Using the initial conditions $a_0=0, b_0=1$ and $a_1=51, b_1=40$, yields

$$a_i=\frac{51\sqrt{3705}}{7410}(61+\sqrt{3705})^i-\frac{51\sqrt{3705}}{7410}(61-\sqrt{3705})^i;$$

$$b_i=(\frac{1235-7\sqrt{3705}}{2470})(61+\sqrt{3705})^i+(\frac{1235+\sqrt{3705}}{2470})(61-\sqrt{3705})^i \tag{4.10}$$

If $r_n=1$, it means that $\Gamma_2^{(n)}$ is without any electrically equivalent transformation. Plugging Eq. (4.10) into Eq. (4.8), we have

$$\tau(\Gamma_2^{(1)})=3\times16^{n-1}r_1^2[(\frac{22477+369\sqrt{3705}}{494})(61+\sqrt{3705})^{n-2}+(\frac{22477-369\sqrt{3705}}{494})(61-\sqrt{3705})^{n-2}]^2, n\geq2. \tag{4.11}$$

When $n=1$ , $\tau(\Gamma_2^{(1)})=3$ which satisfies Eq. (4.11). Therefore, the number of spanning trees in the sequence of the graph $\Gamma_2^{(n)}$ is given by

$$\tau(\Gamma_2^{(1)}) = 3 \times 16^{n-1} r_1^2 [(\frac{22477+369\sqrt{3705}}{494})(61+\sqrt{3705})^{n-2} + (\frac{22477-369\sqrt{3705}}{494})(61-\sqrt{3705})^{n-2}]^2, n \geq 1 \quad (4.12)$$

where

$$r_1 = \frac{(\frac{3713+61\sqrt{3705}}{8})^{n-1}(1251+23\sqrt{3705})+11(21-\sqrt{3705})}{3(\frac{3713+61\sqrt{3705}}{8})^{n-1}(307+4\sqrt{3705})+561}, n \geq 1. \quad (4.13)$$

Inserting Eq. (4.13) into Eq. (4.12) we obtain the desired result. □

## 5. Number of spanning trees in the sequences of $\Gamma_3^{(n)}$ graph

The graph $\Gamma_3^{(n)}$ is defined recursively using the graphs $\Gamma_3^{(1)}$ (triangle or $K_3$) and $\Gamma_3^{(2)}$ as shown in Figure 1. The graph $\Gamma_3^{(n)}$ , n = 3 is obtained by replacing the central triangle in $\Gamma_3^{(2)}$ by a copy of $\Gamma_3^{(2)}$. In general, $\Gamma_3^{(n)}$ is obtained by replacing the central triangle in $\Gamma_3^{(n-1)}$ with $\Gamma_3^{(2)}$. According to this construction, the number of total vertices $|V(\Gamma_3^{(n)})|$ and edges $|E(\Gamma_3^{(n)})|$ are $|V(\Gamma_3^{(n)})| = 9n - 6$ and $|E(\Gamma_3^{(n)})| = 24n - 21$, $n = 1, 2, \ldots$. The average degree of $\Gamma_3^{(n)}$ is in the large $n$ limit which is $\frac{16}{3}$.

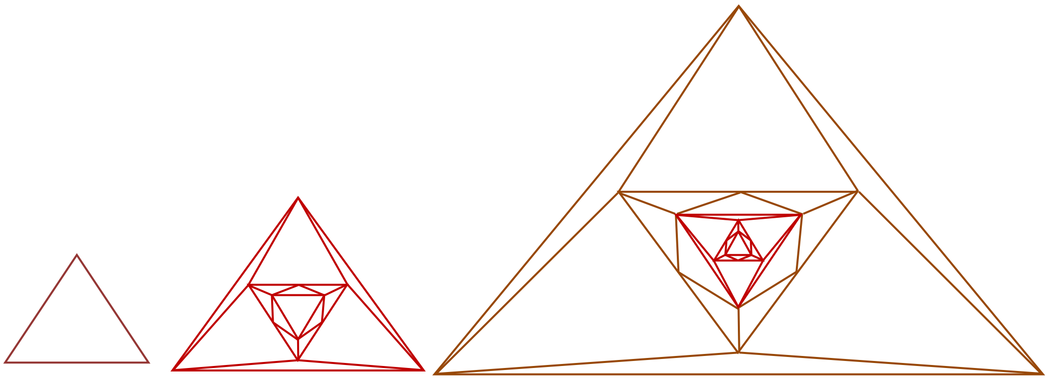
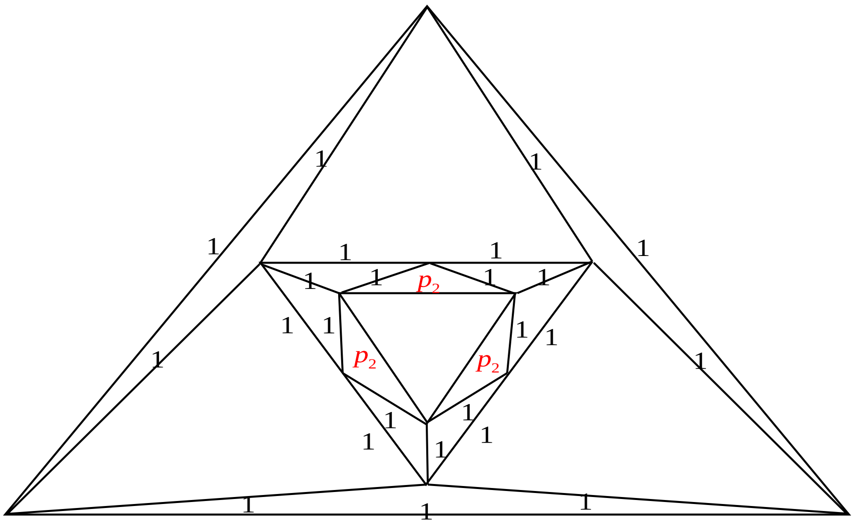
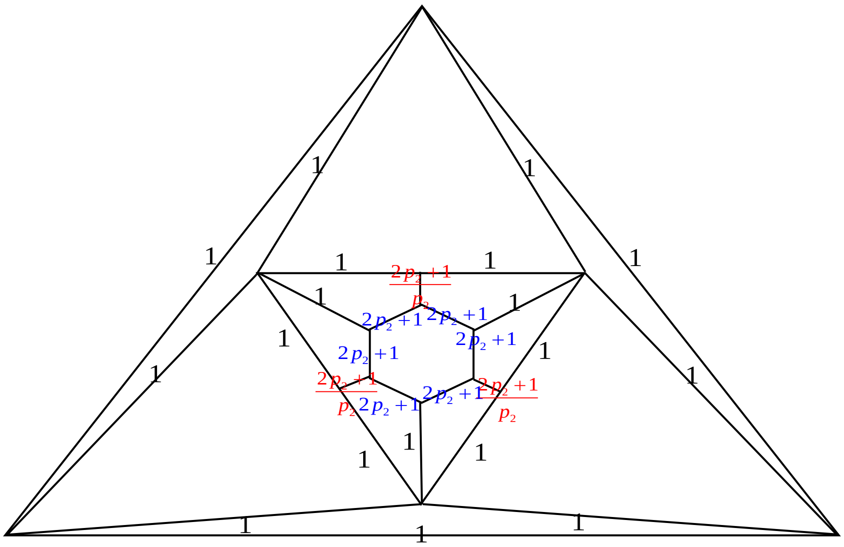



**Fig. 5** Some sequences of the graph $\Gamma_3^{(n)}$

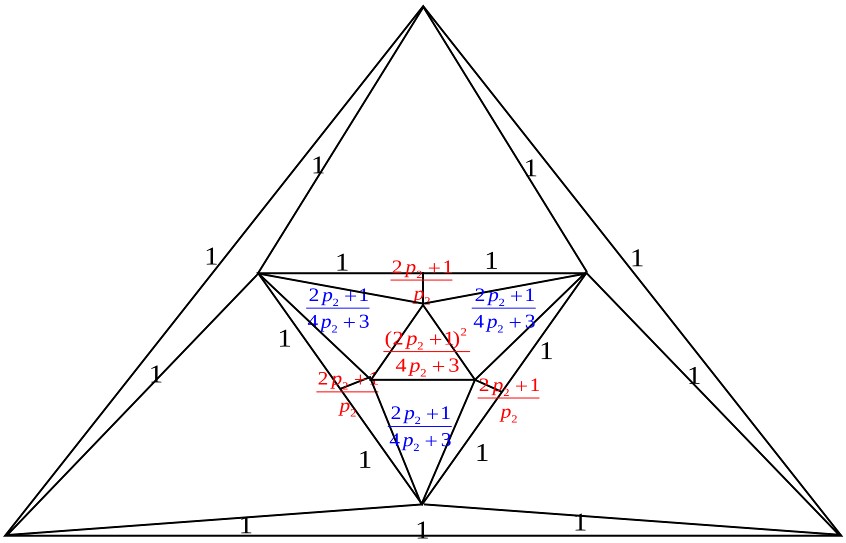
**Theorem 3.** For $n \geq 1$, the number of spanning trees in the sequence of the graph $\Gamma_3^{(n)}$ is given by
$(4^{n-2}((15113 - 768\sqrt{381})(137 + 7\sqrt{381})^n + (137 - 7\sqrt{381})^n(15113 + 768\sqrt{381}))^2(50(7962 + 455\sqrt{381}) - 61(\frac{50}{18719+959\sqrt{381}})^n(857649 + 43939\sqrt{381}))^2)/(1481851875(97625 + 4025\sqrt{381} + 115950^n(18719 + 959\sqrt{381})^{1-n})^2)$

Proof: We use the electrically equivalent transformation to transform $\Gamma_3^{(i)}$ to $\Gamma_3^{(i-1)}$. Fig.6 illustrates the transformation process from $\Gamma_3^{(2)}$ to $\Gamma_3^{(1)}$.

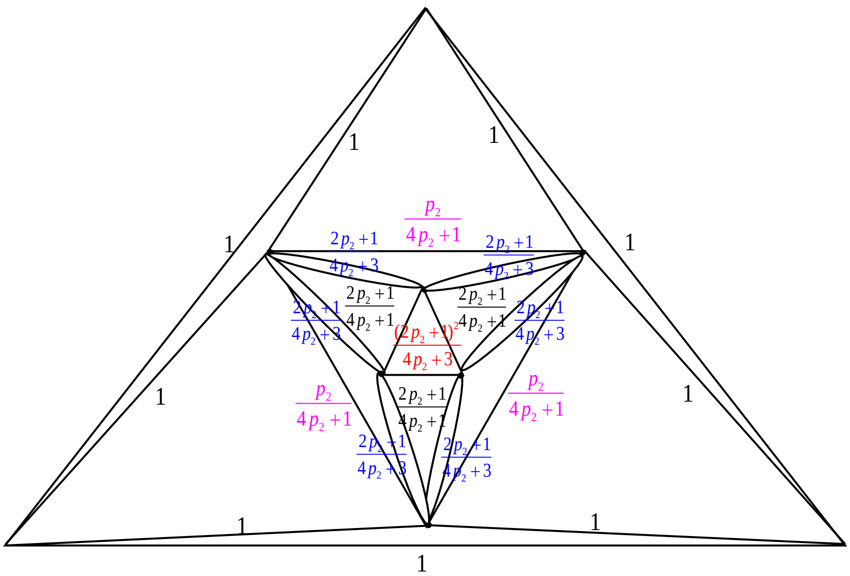



1
2p2+1 / p2
2p2+1

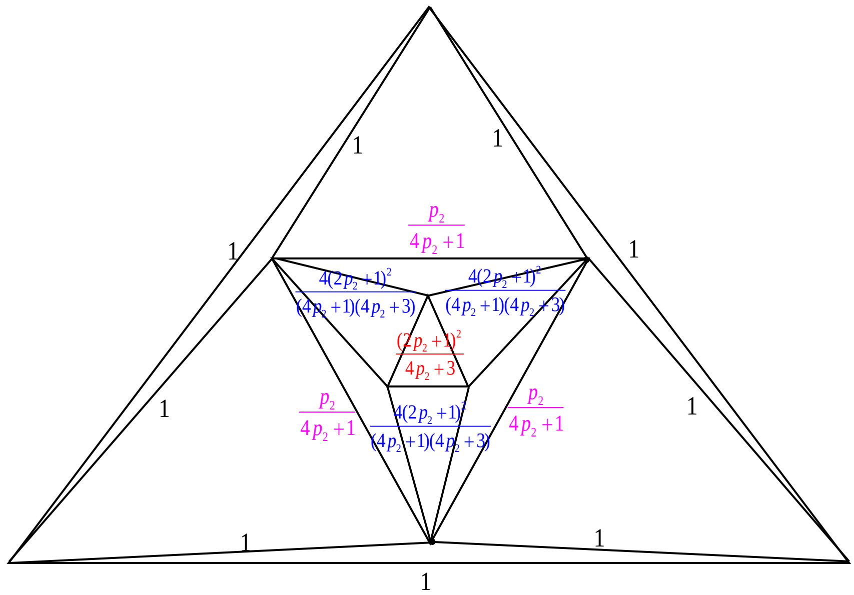
$G_1$


1
2p2+1 / p2
2p2+1 / 4p2+3
(2p2+1)² / 4p2+3

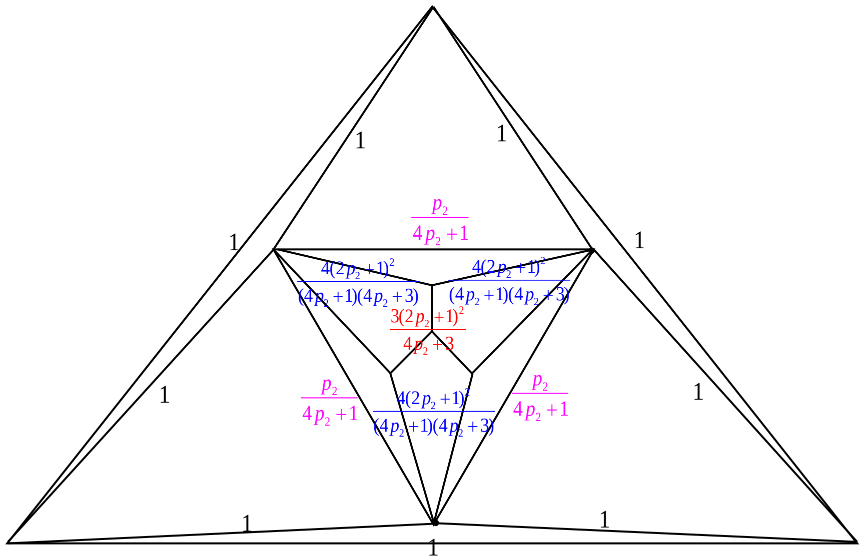
$G_2$


1
p2 / 4p2+1
2p2+1 / 4p2+3
2p2+1 / 4p2+1
(2p2+1)² / 4p2+3

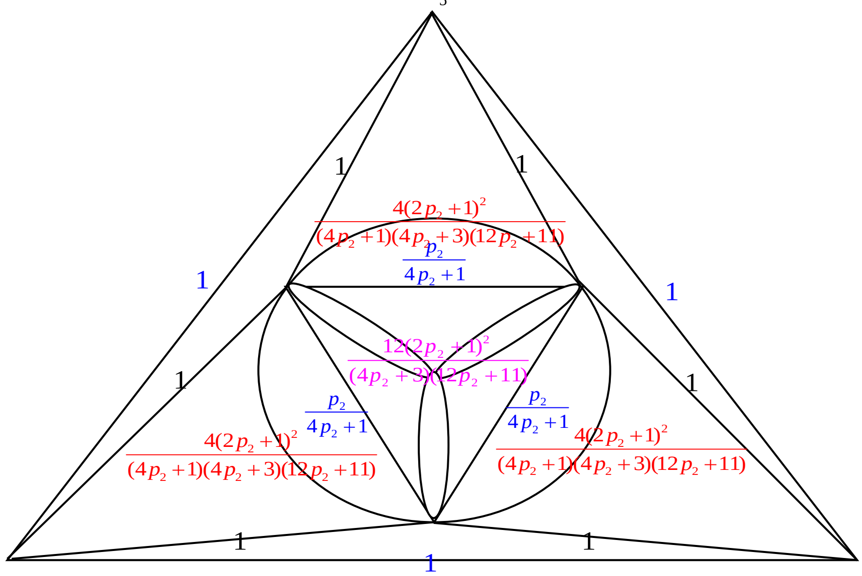
$G_3$


1
$\frac{p_2}{4p_2+1}$
$\frac{4(2p_2+1)^2}{(4p_2+1)(4p_2+3)}$
$\frac{(2p_2+1)^2}{4p_2+3}$

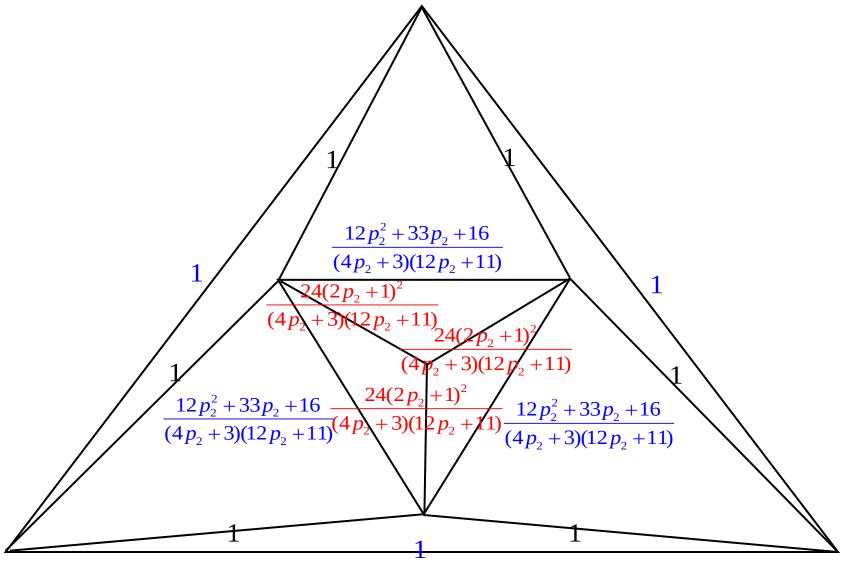
$G_4$


1
$\frac{p_2}{4p_2+1}$
$\frac{4(2p_2+1)^2}{(4p_2+1)(4p_2+3)}$
$\frac{3(2p_2+1)^2}{4p_2+3}$

$G_5$


1
$\frac{4(2p_2+1)^2}{(4p_2+1)(4p_2+3)(12p_2+11)}$
$\frac{p_2}{4p_2+1}$
$\frac{12(2p_2+1)^2}{(4p_2+3)(12p_2+11)}$

$G_6$


1
1
1
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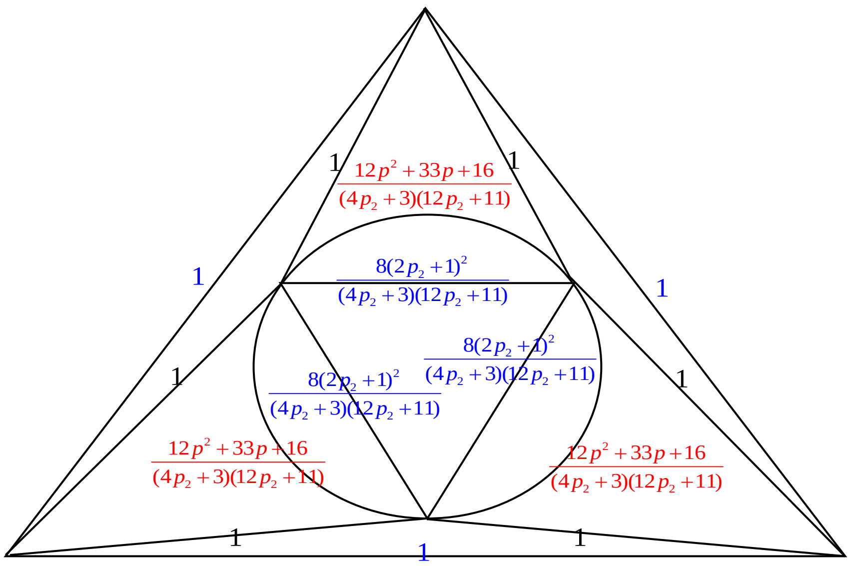
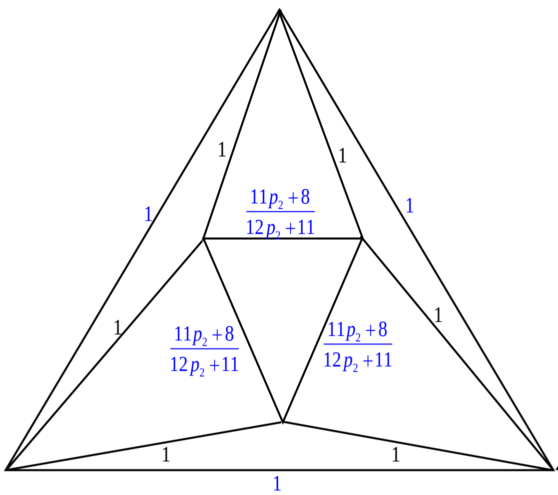
1
1
1
1
1
$\frac{12p_2^2+33p_2+16}{(4p_2+3)(12p_2+11)}$
$\frac{24(2p_2+1)^2}{(4p_2+3)(12p_2+11)}$
$\frac{24(2p_2+1)^2}{(4p_2+3)(12p_2+11)}$
$\frac{24(2p_2+1)^2}{(4p_2+3)(12p_2+11)}$
$\frac{12p_2^2+33p_2+16}{(4p_2+3)(12p_2+11)}$
$\frac{12p_2^2+33p_2+16}{(4p_2+3)(12p_2+11)}$

$G_7$


1
1
1
1
1
1
1
1
1
$\frac{12p^2+33p+16}{(4p_2+3)(12p_2+11)}$
$\frac{8(2p_2+1)^2}{(4p_2+3)(12p_2+11)}$
$\frac{8(2p_2+1)^2}{(4p_2+3)(12p_2+11)}$
$\frac{8(2p_2+1)^2}{(4p_2+3)(12p_2+11)}$
$\frac{12p^2+33p+16}{(4p_2+3)(12p_2+11)}$
$\frac{12p^2+33p+16}{(4p_2+3)(12p_2+11)}$

$G_8$


1
1
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1
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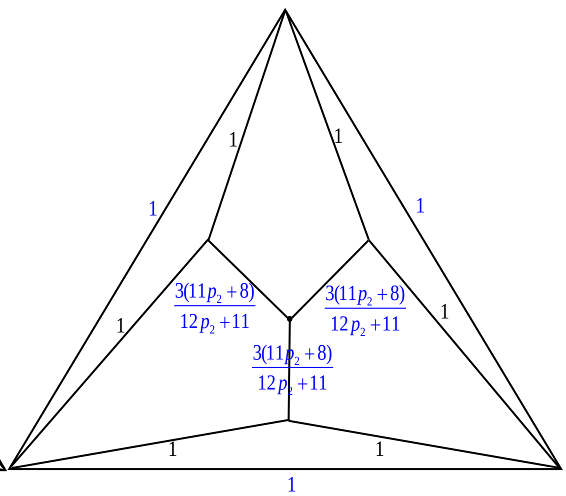
1
1
1
$\frac{11p_2+8}{12p_2+11}$
$\frac{11p_2+8}{12p_2+11}$
$\frac{11p_2+8}{12p_2+11}$

$G_9$


1
1
1
1
1
1
1
1
1
$\frac{3(11p_2+8)}{12p_2+11}$
$\frac{3(11p_2+8)}{12p_2+11}$
$\frac{3(11p_2+8)}{12p_2+11}$

$G_{10}$

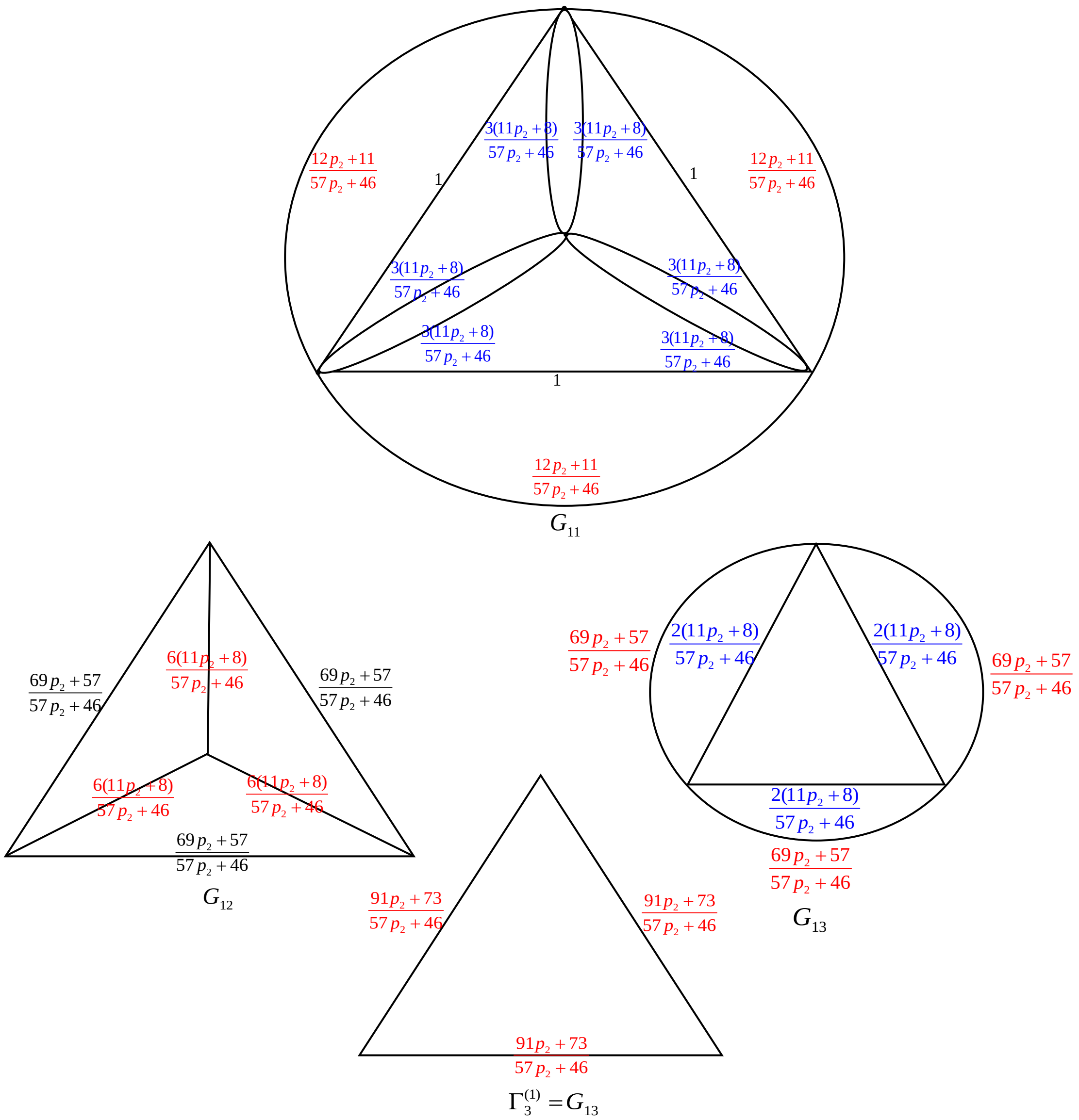



**Fig.6 The** transformations from $\Gamma_3^{(2)}$ to $\Gamma_3^{(1)}$

Using the properties given in section 2, we have the following the transformations:

$\tau(G_1)=[\frac{(2p_2+1)^2}{p_2}]^3\tau(\Gamma_3^{(2)}), \tau(G_2)=[\frac{1}{4p_2+3}]^3\tau(G_1), \tau(G_3)=[\frac{p_2}{4p_2+1}]^3\tau(G_2), \tau(G_4)=\tau(G_3),$
$\tau(G_5)=9[\frac{(2p_2+1)^2}{4p_2+3}]\tau(G_4), \tau(G_6)=[\frac{(4p_2+1)(4p_2+3)}{(2p_2+1)^2(12p_2+11)}]^3\tau(G_5), \tau(G_7)=\tau(G_6), \tau(G_8)=[\frac{(4p_2+3)(12p_2+11)}{72(2p_2+1)^2}]\tau(G_7),$
$\tau(G_9)=\tau(G_8), \tau(G_{10})=9[\frac{11p_2+8}{12p_2+11}]\tau(G_9), \tau(G_{11})=[\frac{12p_2+11}{57p_2+46}]^3\tau(G_{10}), \tau(G_{12})=\tau(G_{11}), \tau(G_{13})=$
$[\frac{57p_2+46}{18(11p_2+8)}]\tau(G_{12})$ and $\tau(\Gamma_3^1)=\tau(G_{13})$.

Combining these fourteen transformations, we have

$$\tau(\Gamma_3^{(2)})=16(57p_2+46)^2\tau(\Gamma_3^{(1)}). \tag{5.1}$$

Further

$$\tau(\Gamma_3^{(n)})=\prod_{i=2}^{n}16(57p_i+46)^2\tau(\Gamma_3^{(1)})=3\times16^{n-1}p_1^2[\prod_{i=2}^{n}(57p_i+46)]^2, \tag{5.2}$$

where $r_{i-1}=\frac{91p_i+73}{57p_i+46}, i=2,3,\ldots,n.$

Its characteristic equation is $57\theta^2-45\theta-73=0$ with roots $\theta_1=\frac{45-7\sqrt{381}}{114}$ and $\theta_2=\frac{45+7\sqrt{381}}{114}$. Subtracting these two roots into both sides of $r_{i-1}=\frac{91p_i+73}{57p_i+46}$, we get

$$r_{i-1}-\frac{45-7\sqrt{381}}{114}=\frac{91p_i+73}{57p_i+46}-\frac{45-7\sqrt{381}}{114}=\frac{(137+7\sqrt{381})(p_i-\frac{45-7\sqrt{381}}{114})}{2(57p_i+46)} \tag{5.3}$$

$$r_{i-1}-\frac{45+7\sqrt{381}}{114}=\frac{91p_i+73}{57p_i+46}-\frac{45+7\sqrt{381}}{114}=\frac{(137-7\sqrt{381})(p_i-\frac{45+7\sqrt{381}}{114})}{2(57p_i+46)} \tag{5.4}$$

Let $q_i = \frac{p_i - \frac{45-7\sqrt{381}}{114}}{p_i - \frac{45+\sqrt{381}}{114}}$. Then by Eqs. (5.3) and (5.4), we get $q_{i-1} = (\frac{18719+959\sqrt{381}}{50})q_i$ and $q_i = (\frac{18719+959\sqrt{381}}{50})^{n-i}q_n$

Therefore $r_i = \frac{(\frac{18719+959\sqrt{381}}{50})^{n-i}(\frac{45+7\sqrt{381}}{114})q_n - (\frac{45-7\sqrt{381}}{114})}{(\frac{18719+959\sqrt{381}}{50})^{n-i}q_n - 1}$.

Thus

$$r_1 = \frac{(\frac{18719+959\sqrt{381}}{50})^{n-1}(\frac{45+7\sqrt{381}}{114})q_n - (\frac{45-7\sqrt{381}}{114})}{(\frac{18719+959\sqrt{381}}{50})^{n-1}q_n - 1}. \tag{5.5}$$

Using the expression $r_{n-1} = \frac{91p_n+73}{57p_n+46}$ and denoting the coefficients of $91r_n + 73$ and $57p_n + 46$ as $a_n$ and $b_n$, we have

$$57r_n + 46 = a_0(91r_n + 73) + b_0(57r_n + 46),$$
$$57r_{n-1} + 46 = \frac{a_1(91r_n + 73) + b_1(57r_n + 46)}{a_0(91r_n + 73) + b_0(57r_n + 46)},$$
$$57r_{n-2} + 46 = \frac{a_2(91r_n + 73) + b_2(57r_n + 46)}{a_1(91r_n + 73) + b_1(57r_n + 46)},$$
$$\vdots$$
$$57r_{n-i} + 46 = \frac{a_i(91r_n+73)+b_i(57r_n+46)}{a_{i-1}(91r_n+73)+b_{i-1}(57r_n+46)}, \tag{5.6}$$
$$57r_{n-(i+1)} + 46 = \frac{a_{i+1}(91r_n+73)+b_{i+1}(57r_n+46)}{a_i(91r_n+73)+b_i(57r_n+46)}, \tag{5.7}$$
$$\vdots$$
$$57r_2 + 46 = \frac{a_{n-2}(91r_n + 73) + b_{n-2}(57r_n + 46)}{a_{n-3}(91r_n + 73) + b_{n-3}(57r_n + 46)},$$

Thus, we obtain

$$\tau(\Gamma_2^{(2)}) = 3 \times 16^{n-1} r_1^2 [a_{n-2}(91r_n + 73) + b_{n-2}(57r_n + 46)]^2 \tag{5.8}$$

where $a_0 = 0, b_0 = 1$ and $a_1 = 57, b_1 = 46$. By the expression $r_{n-1} = \frac{91r_n+73}{57r_n+46}$ and using Eqs. (3.6) and (3.7), we have

$$a_{i+1} = 137a_i - 25a_{i-1};\ b_{i+1} = 137b_i - 25b_{i-1} \tag{5.9}$$

The characteristic equation of Eq. (5.9) is $\varphi^2 - 137\varphi + 25 = 0$ with roots $\varphi_1 = \frac{137+7\sqrt{381}}{2}$ and $\varphi_2 = \frac{137-7\sqrt{381}}{2}$.

The general solutions of Eq. (5.9) are $a_i = h_1\varphi_1^i + h_2\varphi_2^i;\ b_i = k_1\varphi_1^i + k_2\varphi_2^i$.

Using the initial conditions $a_0 = 0, b_0 = 1$ and $a_1 = 57, b_1 = 46$, yields

$$a_i = \frac{19\sqrt{381}}{889}(\frac{137 + 7\sqrt{381}}{2})^i - \frac{19\sqrt{381}}{889}(\frac{137 - 7\sqrt{381}}{2})^i;$$
$$b_i = (\frac{889-15\sqrt{381}}{1778})(\frac{137+7\sqrt{381}}{2})^i + (\frac{889+15\sqrt{381}}{1778})(\frac{137-7\sqrt{381}}{2})^i \tag{5.10}$$

If $r_n = 1$, it means that $\Gamma_3^{(n)}$ is without any electrically equivalent transformation. Plugging Eq. (5.10) into Eq. (5.8), we have

$$\tau(\Gamma_3^{(1)}) = 3 \times 16^{n-1} r_1^2 [(\frac{91567+4687\sqrt{381}}{1778})(\frac{137+7\sqrt{381}}{2})^{n-2} + (\frac{91567-4687\sqrt{381}}{1778})(\frac{137-7\sqrt{381}}{2})^{n-2}]^2, n \geq 2. \tag{5.11}$$

When $n = 1$, $\tau(\Gamma_3^{(1)}) = 3$ which satisfies Eq. (5.11). Therefore, the number of spanning trees in the sequence of the graph $\Gamma_3^{(n)}$ is given by

$$\tau(\Gamma_3^{(1)}) = 3 \times 16^{n-1} r_1^2 [(\frac{91567+4687\sqrt{381}}{1778})(\frac{137+7\sqrt{381}}{2})^{n-2} + (\frac{91567-4687\sqrt{381}}{1778})(\frac{137-7\sqrt{381}}{2})^{n-2}]^2, n \geq 1. \tag{5.12}$$

where

$$r_1 = \frac{(\frac{18719+959\sqrt{381}}{50})^{n-1}(7962+455\sqrt{381})+61(\frac{45-7\sqrt{381}}{2})}{\frac{3}{2}(\frac{18719+959\sqrt{381}}{50})^{n-1}(3905+161\sqrt{381})+3477}, n \geq 1. \tag{5.13}$$

Inserting Eq. (5.13) into Eq.(5.12) we obtain the desired result. □

## 6. Numerical Results

Table1: illustrates some the values of the number of spanning trees in the graphs $\tau(\Gamma_1^{(n)}), \tau(\Gamma_2^{(n)})$ and $\tau(\Gamma_3^{(n)})$.

| $n$ | $\tau(\Gamma_1^{(n)})$ | $\tau(\Gamma_2^{(n)})$ | $\tau(\Gamma_3^{(n)})$ |
|---|---|---|---|
| 1 | 3 | 3 | 3 |
| 2 | 1028376 | 1023168 | 1291008 |
| 3 | 260650372800 | 243223769088 | 386832575232 |
| 4 | 66063400841025024 | 57797705787506688 | 115857765449084928 |
| 5 | 16744165306881362178048 | 13734568487453215162368 | 34699802317209927155712 |
| 6 | 4243909157792864879554166784 | 3263769190981487290774192128 | 10392711055982113862055886848 |

## 7. Spanning Tree Entropy

After having explicit Formulas for the number of spanning trees of the sequence of the six graphs $\Gamma_1^{(n)}, \Gamma_2^{(n)}$ and $\Gamma_3^{(n)}$, we can calculate its spanning tree entropy $Z$ which is a finite number and a very interesting quantity characterizing the network structure, defined as in [20] as: For a graph $G$,

$$Z(G) = \lim_{n\to\infty} \frac{\ln \tau(G)}{|V(G)|}. \tag{7.1}$$

$$Z(\Gamma_2^{(n)}) = \frac{1}{9}\left(ln[8] + 2\,ln\left[89 + 12\sqrt{55}\right]\right) = 1.3825495018$$

$$Z(\Gamma_2^{(n)}) = \frac{2}{9}[ln\,4\,(61 + \sqrt{3705})] = 1.3753863752.$$

$$Z(\Gamma_3^{(n)}) = \frac{1}{9}(ln[4] + 2\,ln[137 + 7\sqrt{381}]) = 1.4010979187.$$

Now we compare the value of entropy in our graphs with other graphs. The entropy of the graph $\Gamma_3^{(n)}$ is larger than the entropy of the graph $\Gamma_1^{(n)}$ and the graph $\Gamma_2^{(n)}$ . In addition the entropy of our three families $\Gamma_1^{(n)}, \Gamma_2^{(n)}$ and $\Gamma_3^{(n)}$ which have average degree $^{16}/_{3}$ is larger than the entropy of fractal scale free lattice [21] which has the entropy1.040 of average degree 4 and the entropy of apollonian graph [22] which has the entropy 1.3540 of the same average degree 5.

## 8. Conclusions

In this paper, we calculate the number of spanning trees in the sequences of some graphs generated by triangle graph using electrically equivalent transformations. The feature of this technique lies in the parry of strenuous computation of Laplacian spectra that is prerequisite for a generic method for determining spanning trees. In addition, our results have shown that the entropy is related to the average degree of the graph.

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