Complexity of new families of graphs that have the same average degree and their Entropies

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Abstract

The number of spanning trees is an important quantity characterizing the reliability of a network(graph). Generally, the number of spanning trees in a network can be obtained by directly calculating a related determinant corresponding to the network. However, for a large network, evaluating the relevant determinant is intractable. In this paper, we investigated the number of spanning trees in three sequences of families of graphs of the same average degree $\frac{16}{3}$. We used the electrically equivalent transformations and rules of weighted generating function which avoids the laborious computation of the determinant for counting the number of spanning trees. Finally, we determined the entropy of our studied graphs.

Keywords: Number of spanning trees; Electrically equivalent transformations; Entropy.

Mathematics Subject Classification: 05C30, 05C50, 05C63.

1. Introduction

In graph theory, determining the number of spanning trees in a graph is one of the most extensively researched issues.

An $(n-1)-$ edge subgraph of a connected graph G with n vertices is a spanning tree of that graph G. In graph theory, the number of spanning trees in a graph represented as $\tau(G)$ is also known as the complexity of G and is a significant and extensively researched quantity that is used in many different applications. Counting the number of Eulerian circuits in a graph [1], recalling specific chemical isomers [2], and network dependability [3] are the application domains that people remember the most. Counting spanning trees is a crucial component in many approaches to network reliability estimation, computation, and bounding [4]. In a network represented by a graph, communication amongst each network node of the network implies that the graph must contain a spanning tree and, thus, maximizing the number of spanning trees is a way of maximizing reliability.

There exist various methods for finding this number. Kirchhoff's matrix tree theorem named after Gustav Kirchhoff [5] is a theorem about the number of spanning trees in a graph, showing that this number can be computed in polynomial time from the determinant of a submatrix of the Laplacian matrix of the graph; specifically, the number is equal to any cofactor of the Laplacian matrix.

Another method to count the complexity of a graph is using Laplacian eigenvalues. Let G be a connected graph with k vertices. Kelmans and Chelnoknov [6] derived the following formula:

$$\tau(G)=\frac{1}{k}\prod_{i=1}^{k-1}\mu_i. \tag{1.1}$$

Where $k=\mu_1\geq\mu_2\geq\cdots\geq\mu_k=0$ are the eigenvalues of the Kirchhoff matrix L.

Degenerating the graph through successive elimination of contraction of its edges represents the core of another way to compute the complexity of a graph [7,8]. If $G=(V,E)$ is a multigraph with $e\in E$, then $G.e$ is the graph obtained from G by contracting the degree until its endpoints are a single vertex. The formula for computing the number of spanning trees of a multigraph G is given by:

$$\tau(G)=\tau(G-e)+\tau(G.e) \tag{1.2}$$

This formula is beautiful but not practically useful (grows exponentially with the size of the graph- may be as many as $2^{|E(G)|}$terms. For a summary of further results for calculating the number of spanning trees of graphs, see [9, 10,11,12,13,14,15].

2. Electrically equivalent transformations

An edge-weighted graph, whose weights represent the conductance of the corresponding edges, may be thought of as an electrical network, which is why Kirchhoff was motivated to research electrical networks. The quotient of the (weighted) number of spanning trees and the (weighted) number of so-called thickets—that is, spanning forests with exactly two components and the characteristic that each component contains precisely one of the vertices u,v can be used to express the effect conductance between two vertices u,v [16,17,18,19,20]. The impact of a few basic modifications on the quantity of spanning trees is listed below. The weighted number of spanning trees G is indicated by $\tau(G)$ and let G be an edge weighted graph and G' be the associated electrically equivalent graph.

• **Parallel edges:** When two parallel edges in G, each with conductances u and v, are merged into a single edge in G' with a conductance of $u+v$, the count of spanning trees, $\tau(G')$, remains unchanged compared to $\tau(G)$.

• **Serial edges:** If two serial edges in G, with conductances u and v, are combined into a single edge in G' with a conductance of $uv/(u+v)$, then $\tau(G')$ can be calculated as $(1/(u+v)$ multiplied by $\tau(G)$.

• **Δ-Y Transformation:** When a triangle in G, with conductances x, y and z is transformed into an electrically equivalent star graph in G' with conductances $x = (uv + vw + wu)/u, y = (uv + vw + wu)/v$, and $z = (uv + vw + wu)/w$, the count of spanning trees in $G', \tau(G')$, can be determined as $(uv + vw + wu)^2 / uvw$ multiplied by $\tau(G)$.
• **Y-Δ Transformation:** If a star graph in G, with conductances u, v and w, is converted into an electrically equivalent triangle in G' with conductances $x = vw/(u + v + w), y = uw/(u + v + w)$ and $z = uv/(u + v + w)$, then $\tau(G')$ is given by $1/(u + v + w)$ multiplied by $\tau(G)$.

In mathematics, it is common to derive new structures from existing ones. This principle extends to graphs, where numerous new graphs can be generated from a given set. In this study, we determine the complexity for four novel types of graphs of the same average degree we named it $\Pi_1^{(n)}, \Pi_2^{(n)}$ and $\Pi_3^{(n)}$ respectively.

3. Number of spanning trees in the sequences of $\Pi_1^{(n)}$ graph

The graph $\Pi_1^{(n)}$ is defined recursively using the graphs $\Pi_1^{(1)}$ (triangle or K_3) and $\Pi_1^{(2)}$ as shown in Figure 1. The graph $\Pi_1^{(1)}$, n = 3 is obtained by replacing the central triangle in $\Pi_1^{(2)}$ by a copy of $\Pi_1^{(2)}$. In general, $\Pi_1^{(n)}$ is obtained by replacing the central triangle in $\Pi_1^{(n-1)}$ with $\Pi_1^{(2)}$. According to this construction, the number of total vertices $\left|V(\Pi_1^{(n)})\right|$ and edges $\left|E(\Pi_1^{(n)})\right|$ are $\left|V(\Pi_1^{(n)})\right| = 9n - 6$ and $\left|E(\Pi_1^{(n)})\right| = 24n - 21$, n = 1, 2, ... The average degree of $\Pi_1^{(n)}$ is in the large n limit which is $\frac{16}{3}$.

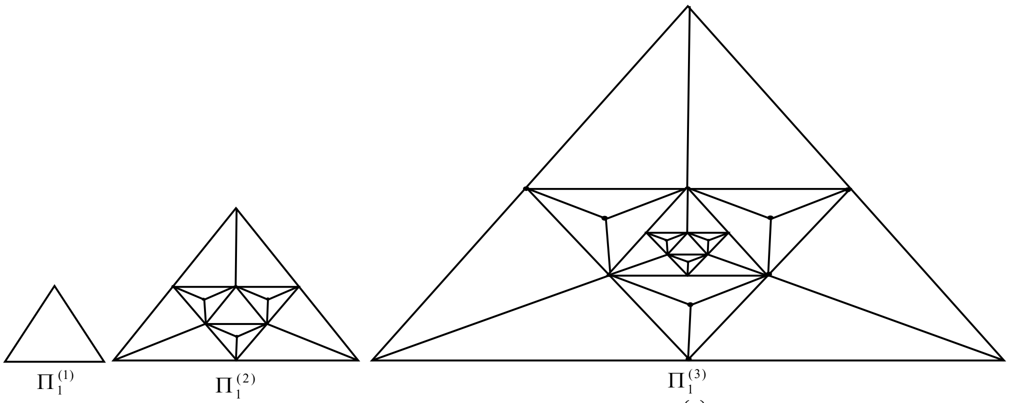
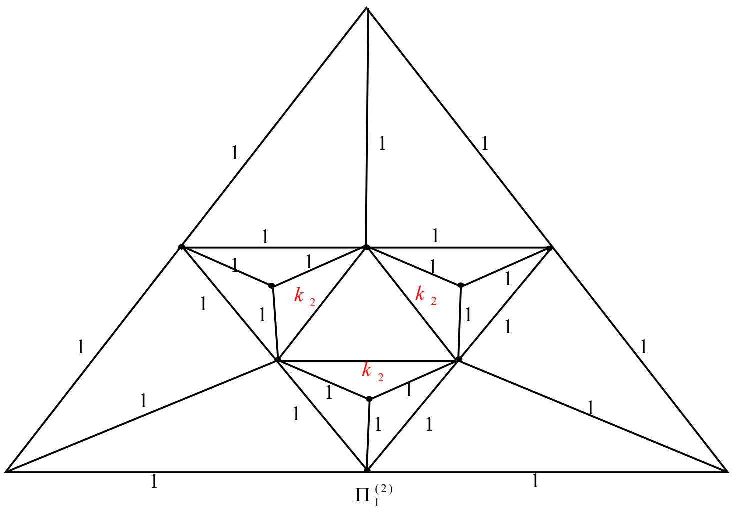


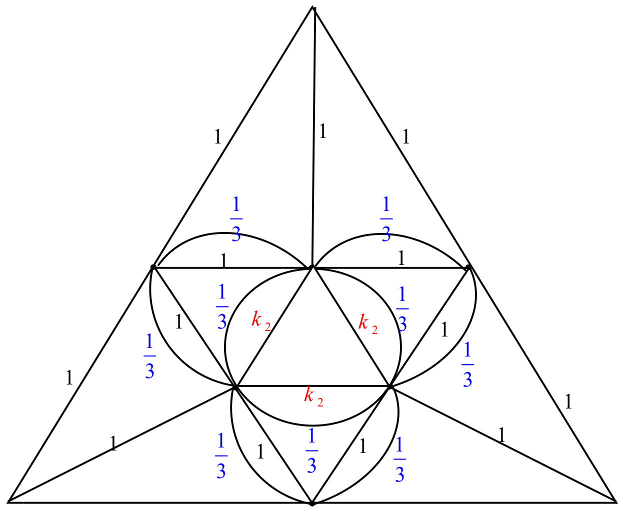
Fig.1 Some sequences of the graph $\Pi_1^{(n)}$

Theorem 1. For $n \geq 1$, the number of spanning trees in the sequence of the graph $\Pi_1^{(n)}$ is given by

$$\frac{2^{-7-n} \times 3^{-8+3n} \times 5^{-1+n}((33-3\sqrt{105})^n(3+\sqrt{105})-(-3+\sqrt{105})(3(11\ \ \sqrt{105}))^n)^2(7+3\sqrt{105}+2^{4-3n}(91+9\sqrt{105})(113+\ \ \sqrt{105})^{-1+n})^2)}{(28+72^{-2-3n}(-19+\sqrt{105})(113+\ \ \sqrt{105})^n)^2}$$

Proof: We use the electrically equivalent transformation to transform $\Pi_1^{(i)}$ to $\Pi_1^{(i-1)}$. Fig.2 illustrates the transformation process from $\Pi_1^{(2)}$ to $\Pi_1^{(1)}$.




1
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k_2
k_2
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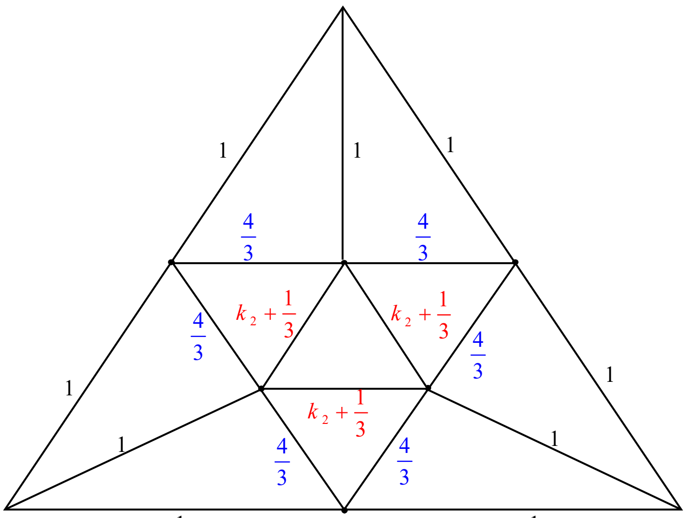
G_1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4/3
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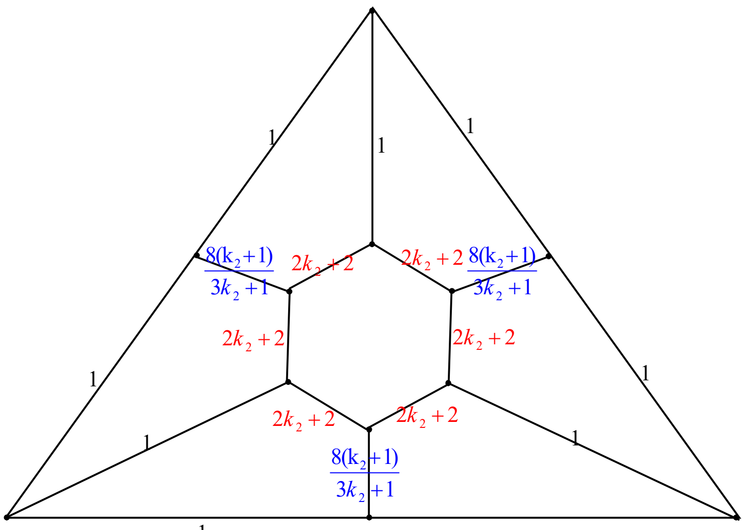
G_2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$\frac{8(k_2+1)}{3k_2+1}$
$\frac{8(k_2+1)}{3k_2+1}$
$2k_2+2$
$2k_2+2$
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1
1

G_3

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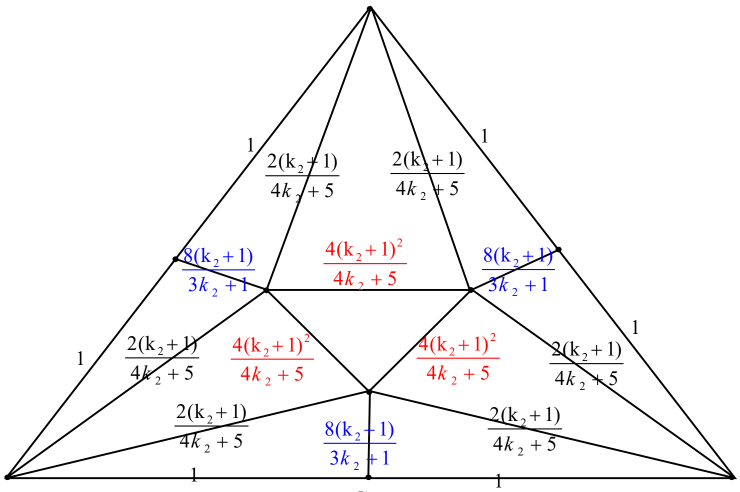
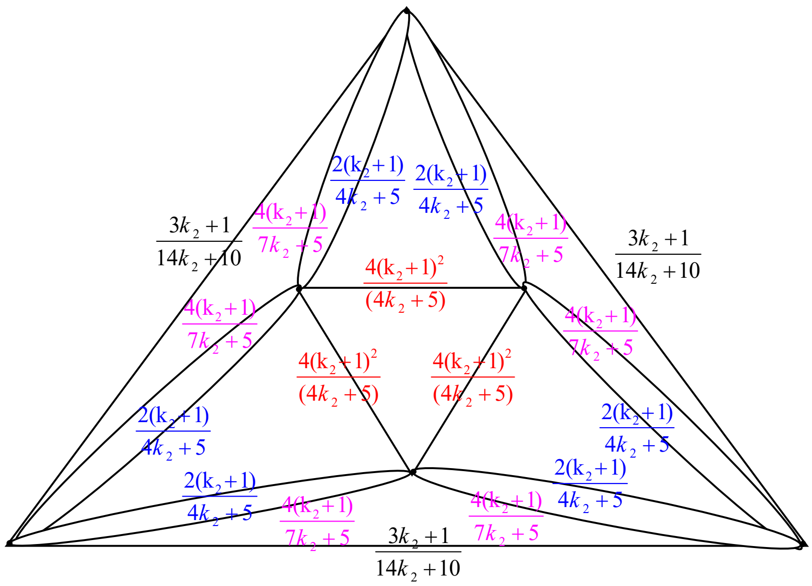
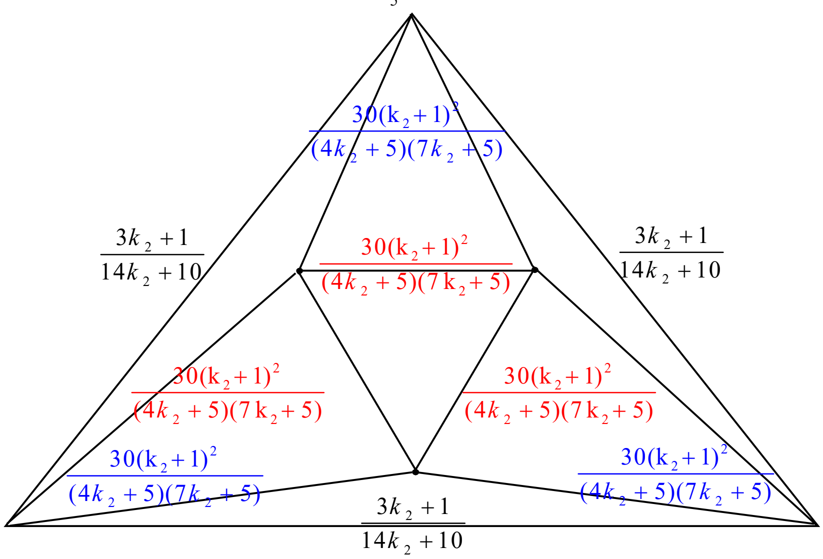


G_4



G_5



G_6

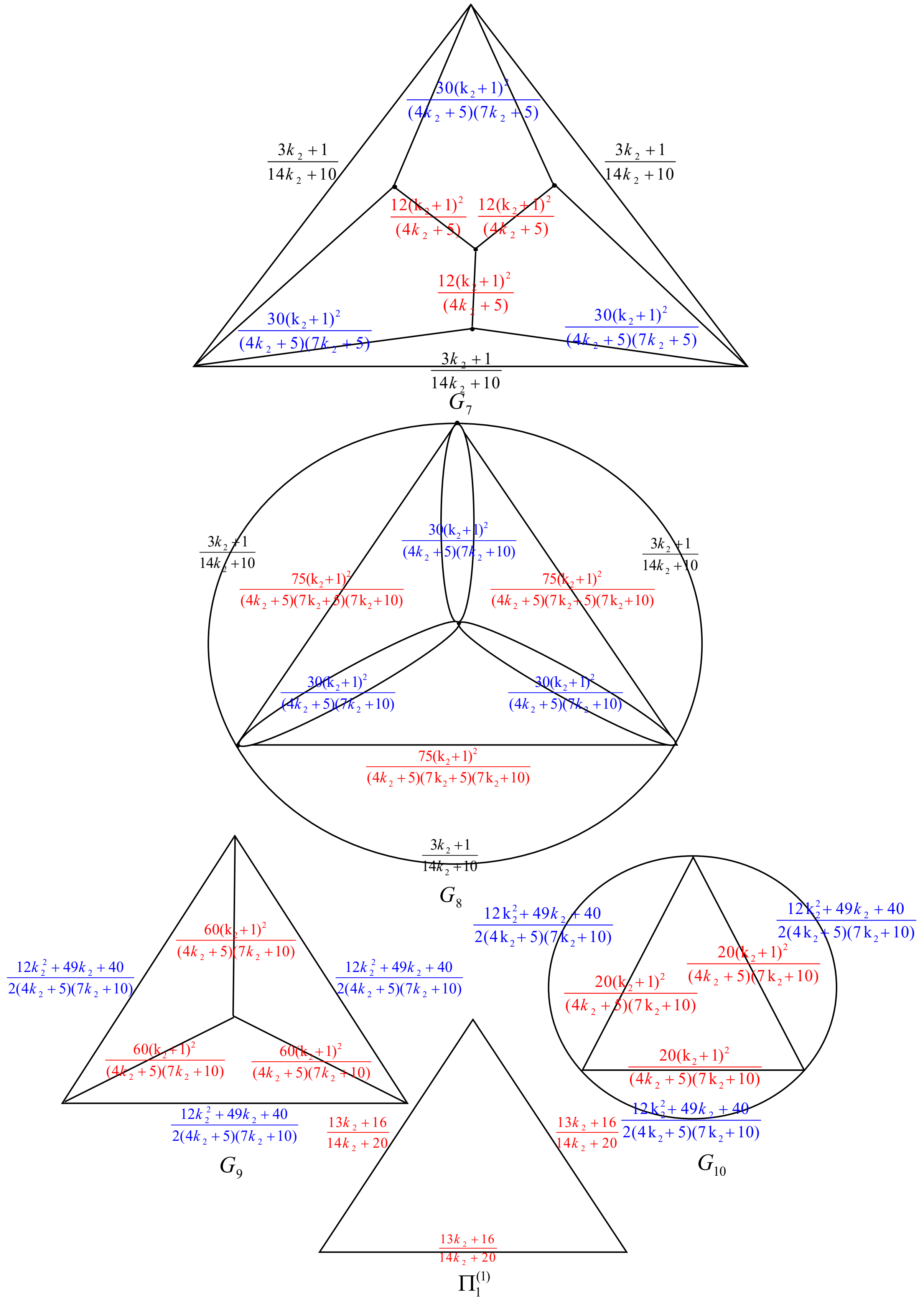


Fig.2 The transformations from $\Pi_1^{(2)}$ to $\Pi_1^{(1)}$

Using the properties given in section 2, we have the following the transformations:
$\tau(G_1) = [\frac{1}{3}]^3\tau(\Pi_1^{(2)}), \tau(G_2) = \tau(G_1), \tau(G_3) = [\frac{12(k_2+1)^2}{3k_2+1}]^3\tau(G_2), \tau(G_4) = [\frac{1}{4k_2+5}]^3\tau(G_3), \tau(G_5) =$
$[\frac{3k_2+1}{14k_2+10}]^3\tau(G_4), \tau(G_6) = \tau(G_5), \tau(G_7) = [\frac{36(k_2+1)^2}{4k_2+5}]^3\tau(G_6), \tau(G_8) = [\frac{(4k_2+5)(7k_2+5)}{12(k_2+1)^2(7k_2+10)}]^3\tau(G_7), \tau(G_9) = \tau(G_8)$
$\tau(G_{10}) = [\frac{(4k_2+5)(7k_2+10)}{180(k_2+1)^2}]\tau(G_9)$, and $\tau(\Pi_1^{(1)}) = \tau(G_{10})$.
Combining these eleven transformations, we have:

$$\tau(\Pi_1^{(2)}) = 270(14k_2+00)^2\tau(\Pi_1^{(1)}). \tag{3.1}$$

Further

$$\tau(\Pi_1^{(n)}) = \prod_{i=2}^{n} 270(14k_i+00)^2\tau(\Pi_1^{(1)}) = 3\times 270^{n-1}k_1^2[\prod_{i=2}^{n}(14k_i+20)]^2, \tag{3.2}$$

where $k_{i-1} = \frac{13\ \ _i+16}{14k_i+20}, i = 2,3,\dots,n$.

Its characteristic equation is $14\alpha^2 + 7\alpha - 16 = 0$ with roots $\alpha_1 = \frac{-7-\sqrt{105}}{28}$ and $\alpha_2 = \frac{-7+3\sqrt{105}}{28}$. Subtracting these two roots into both sides of $k_{i-1} = \frac{13k_i+16}{14k_i+20}$, we get

$$k_{i-1} - \frac{-7-3\sqrt{1055}}{28} = \frac{13k_i+16}{14\ \ _i+2} + \frac{7+3\sqrt{105}}{28} = \frac{(33+3\sqrt{105})(k_i+\frac{7+3\sqrt{105}}{28})}{2(14k_i+2\ \ )} \tag{3.3}$$

$$k_{i-1} - \frac{-7+3\sqrt{1055}}{28} = \frac{13k_i+16}{14k_i+20} + \frac{7-3\sqrt{105}}{28} = \frac{(33-3\sqrt{105})(k_i+\frac{7-3\sqrt{105}}{28})}{2(14k_i+20)} \tag{3.4}$$

Let $h_i = \frac{k_i+\frac{7+3\sqrt{105}}{28}}{k_i+\frac{7-3\sqrt{105}}{28}}$.Then by Eqs. (3.3) and (3.4), we get $h_{i-1} = (\frac{113+11\sqrt{105}}{8})h_i$ and $h_i = (\frac{113+11\sqrt{105}}{8})^{n-i}h_n$.

Therefore $k_i = \frac{(\frac{113+11\sqrt{105}}{8})^{n-i}(\frac{-7+3\sqrt{105}}{28})h_n+(\frac{7+3\sqrt{55}}{28})}{(\frac{113+11\sqrt{105}}{8})^{n-i}h_n-1}$.

Thus

$$k_1 = \frac{(\frac{113+11\sqrt{105}}{8})^{n-1}(\frac{-7+3\sqrt{105}}{28})h_n+(\frac{7+3\sqrt{55}}{28})}{(\frac{113+1\ \ \sqrt{105}}{8})^{n-1}h_n-1}.. \tag{3.5}$$

Using the expression $k_{n-1} = \frac{13\ \ _n+16}{14\ \ _n+20}$ and denoting the coefficients of $13k_n + 16$ and $14k_n + 20$ as f_n and g_n , we have

$$14k_n + 20 = f_0(13k_n + 16) + g_0(14k_n + 20),$$
$$14k_{n-1} + 20 = \frac{f_1(13k_n + 16) + g_1(14k_n + 20)}{f_0(13k_n + 16) + g_0(14k_n + 20)},$$
$$14k_{n-2} + 20 = \frac{f_2(13k_n + 16) + g_2(14k_n + 20)}{f_1(13k_n + 16) + g_1(14k_n + 20)},$$
$$\vdots$$

$$14k_{n-i} + 20 = \frac{f_i(13k_n+16)+g_i(14k_n+20)}{f_{i-1}(13k_n+16)+g_{i-1}(14k_n+20)}, \tag{3.6}$$

$$14k_{n-(i+1)} + 20 = \frac{f_{i+1}(13k_n+16)+g_{i+1}(14k_n+20)}{f_i(13\ \ _n+1\ \ )+g_i(14k_n+20)}, \tag{3.7}$$

$$\vdots$$

$$14k_2 + 20 = \frac{f_{n-2}(13k_n + 16) + g_{n-2}(14k_n + 20)}{f_{n-3}(13k_n + 16) + g_{n-3}(14k_n + 20)},$$

Thus, we obtain

$$\tau(\Pi_1^{(n)}) = 3\times 270^{n-1}k_1^2[f_{n-2}(13k_n + 16) + g_{n-2}(14k_n + 20)]^2 \tag{3.8}$$

where $f_0 = 0, g_0 = 1$ and $f_1 = 14, g_1 = 20$. By the expression $k_{n-1} = \frac{13k_n+16}{14\ \ _n+20}$ and using Eqs. (3.6) and (3.7), we have

$$f_{i+1} = 33f_i - 36f_{i-1}; g_{i+1} = 33g_i - 36g_{i-1} \tag{3.9}$$

The characteristic equation of Eq.(3.9) is $\beta^2 - 33\beta + 36 = 0$ with roots $\beta_1 = \frac{33+3\sqrt{105}}{2}$ and $\beta_1 = \frac{33-3\sqrt{105}}{2}$. The general solutions of Eq. (3.9) are
$f_i = a_1\beta_1^i + a_2\beta_2^i; g_i = b_1\beta_1^i + b_2\beta_2^i$.
Using the initial conditions $a_0 = 0, b_0 = 1$ and $f_1 = 14, g_1 = 20$, yields

$$f_i = \frac{2\sqrt{105}}{45}(\frac{33+3\sqrt{105}}{2})^i - \frac{2\sqrt{105}}{45}(\frac{33-3\sqrt{105}}{2})^i; g_i = (\frac{45+\sqrt{105}}{90})(\frac{33+3\sqrt{105}}{2})^i + (\frac{45\ \ \sqrt{105}}{90})(\frac{33-3\sqrt{105}}{2})^i \tag{3.10}$$

If $k_n = 1$, it means that $\Pi_1^{(n)}$ is without any electrically equivalent transformation. Plugging Eq. (3.10) into Eq. (3.8), we have

$$\tau(\Pi_1^{(n)}) = 3\times 270^{n-1}k_1^2[(\frac{51+5\sqrt{105}}{3})(\frac{33+3\sqrt{105}}{2})^{n-2} + (\frac{51-5\sqrt{105}}{3})(\frac{33-\ \ \sqrt{105}}{2})^{n-2}]^2, n \geq 2. \tag{3.11}$$

When $n = 1$, $\tau(\Pi_1^{(1)}) = 3$ which satisfies Eq. (3.11). Therefore, the number of spanning trees in the sequence of the graph $\Pi_1^{(n)}$ is given by

$$\tau(\Pi_1^{(n)}) = 3\times 270^{n-1}k_1^2[(\frac{51+5\sqrt{105}}{3})(\frac{33+3\sqrt{105}}{2})^{n-2} + (\frac{51-5\sqrt{105}}{3})(\frac{33-3\sqrt{105}}{2})^{n-2}]^2, n \geq 1. \tag{3.12}$$

Where

$$k_1 = \frac{2(\frac{113+11\sqrt{105}}{8})^{n-1}(91+9\sqrt{105})+(7+3\sqrt{105})}{7(\frac{113+\sqrt{105}}{8})^{n-1}(31+3\sqrt{105})-28}, n \geq 1. \qquad (3.13)$$

Inserting Eq. (3.13) into Eq.(3.12) we obtain the desired result. □

4. Number of spanning trees in the sequences of $\Pi_2^{(n)}$ graph

The graph $\Pi_2^{(n)}$ is defined recursively using the graphs $\Pi_2^{(1)}$ (triangle or K_3) and $\Pi_2^{(2)}$ as shown in Figure 1. The graph $\Pi_2^{(1)}$, n = 3 is obtained by replacing the central triangle in $\Pi_2^{(2)}$ by a copy of $\Pi_2^{(2)}$. In general, $\Pi_2^{(n)}$is obtained by replacing the central triangle in $\Pi_2^{(n-1)}$ with $\Pi_2^{(2)}$. According to this construction, the number of total vertices$\left|V(\Pi_2^{(n)})\right|$ |and edges $\left|E(\Pi_2^{(n)})\right|$are $\left|V(\Pi_2^{(n)})\right| = 9n - 6$ and $\left|E(\Pi_2^{(n)})\right| = 24n - 21$, n = 1, 2, ... The average degree of $\Pi_2^{(n)}$ is in the large n limit which is $\frac{16}{3}$.

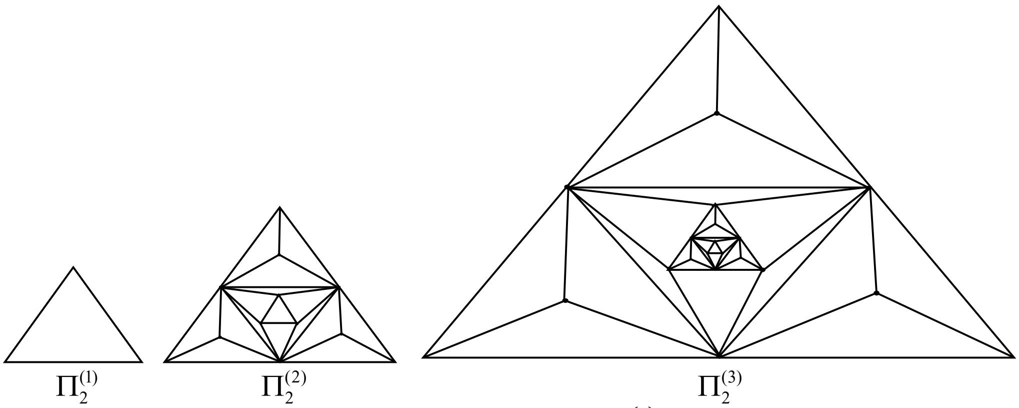
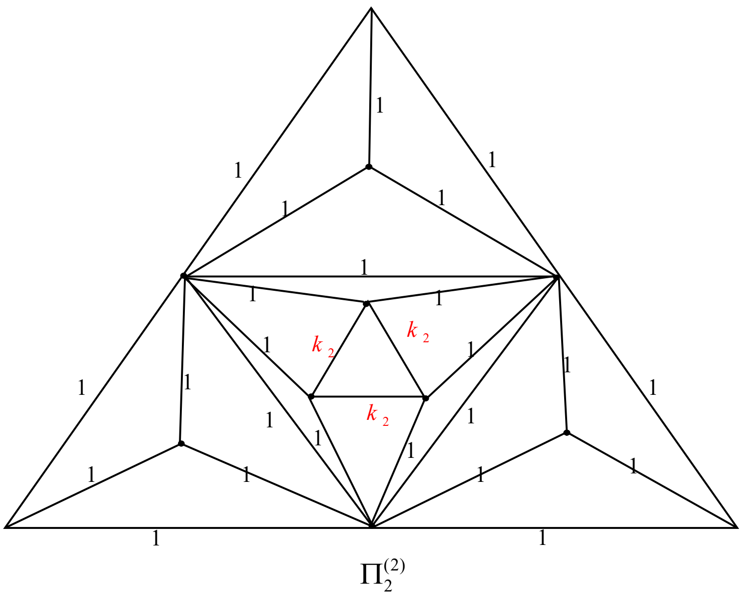


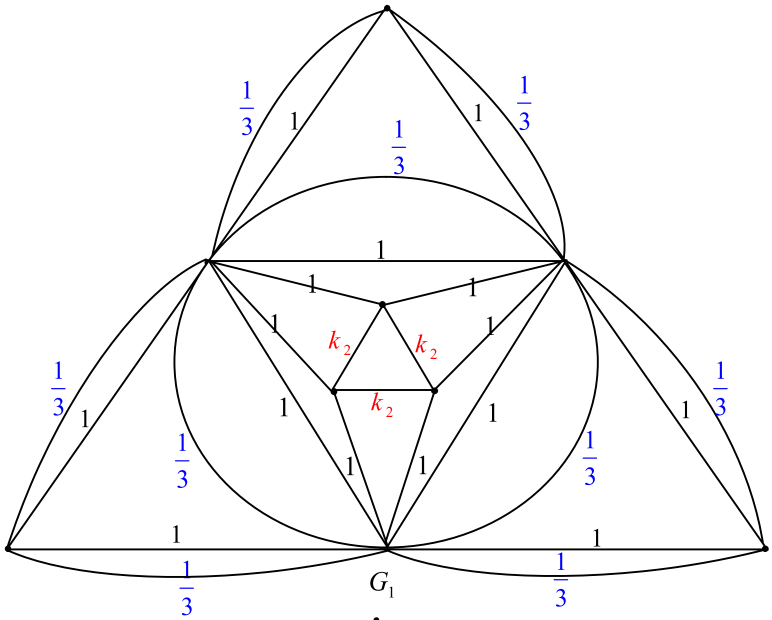
Fig. 3 Some sequences of the graph $\Pi_2^{(n)}$

Theorem 2. For $n \geq 1$, the number of spanning trees in the sequence of the graph $\Pi_2^{(n)}$ is given by

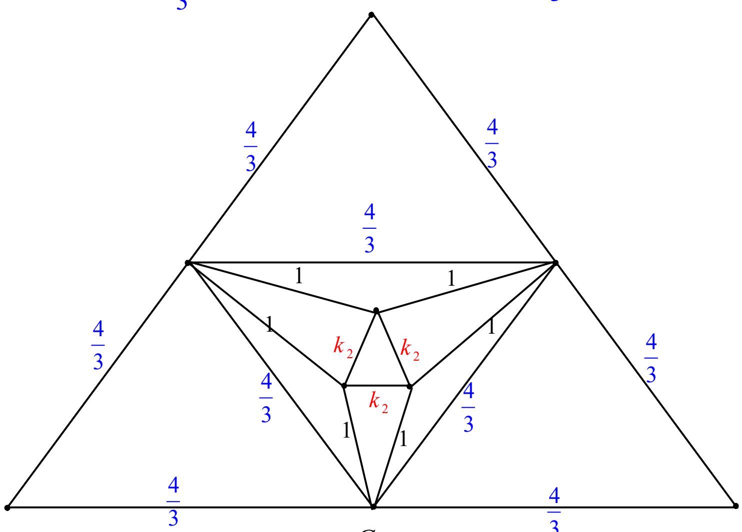
$$\frac{(3\times 4^{n-6}((2695-269\sqrt{105})(113+11\sqrt{105})^n+(113-\sqrt{105})^n(2695+269\sqrt{105}))^2(\frac{23}{2}(-15+11\sqrt{105})+(6525+649\sqrt{105})(\frac{32}{12737+1243\sqrt{105}})^{1-n})^2)}{(148225(1794-{}^{4-5n}(5431+51\sqrt{105})(12737+1243\sqrt{105})^{n-1})^2)}$$

Proof: We use the electrically equivalent transformation to transform $\Pi_2^{(i)}$ to $\Pi_2^{(i-1)}$. Fig.4 illustrates the transformation process from $\Pi_2^{(2)}$ to $\Pi_2^{(1)}$.

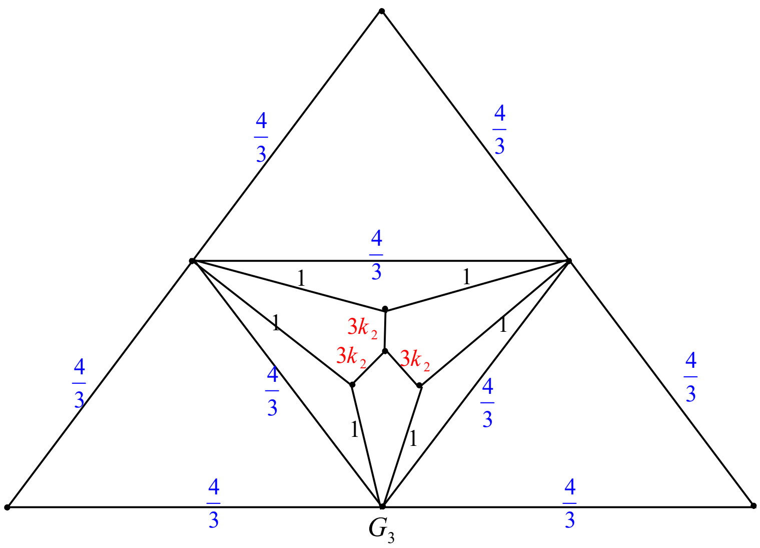




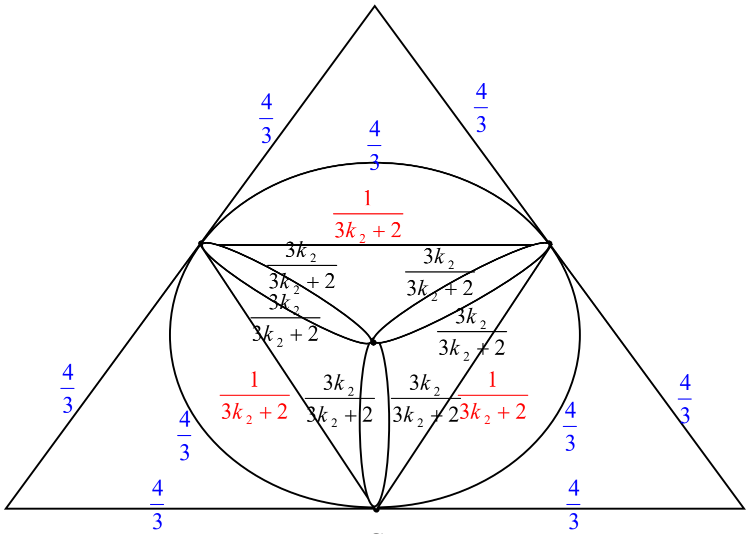
G_1



G_2



G_3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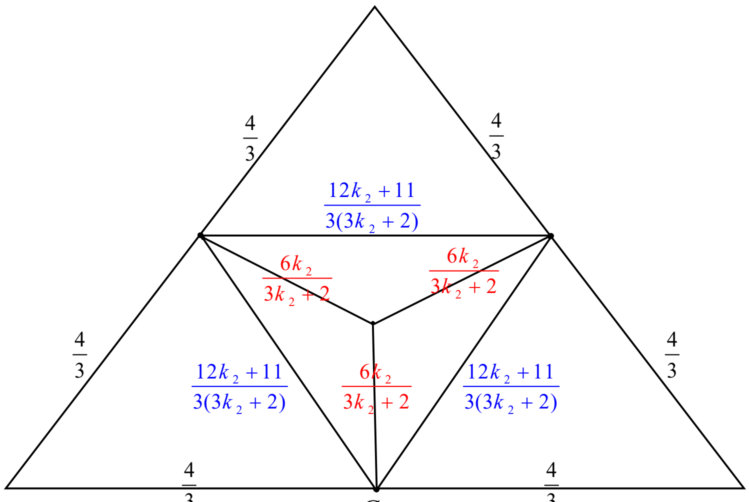
$\frac{1}{3k_2+2}$
$\frac{3k_2}{3k_2+2}$

G_4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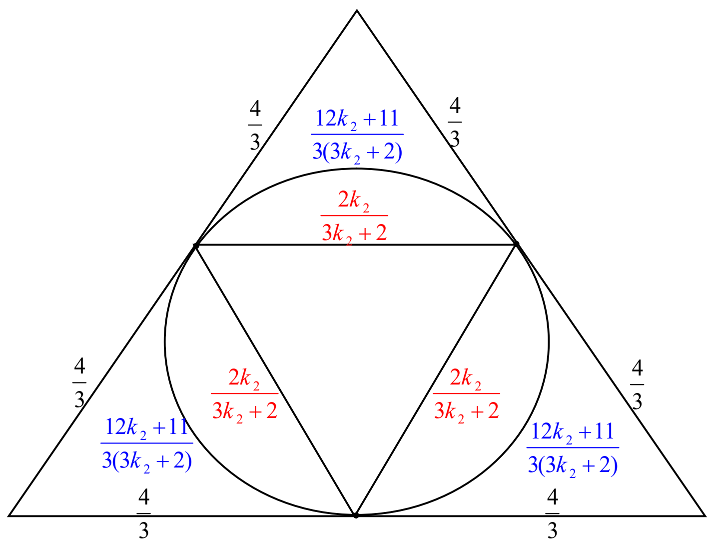
$\frac{6k_2}{3k_2+2}$

G_5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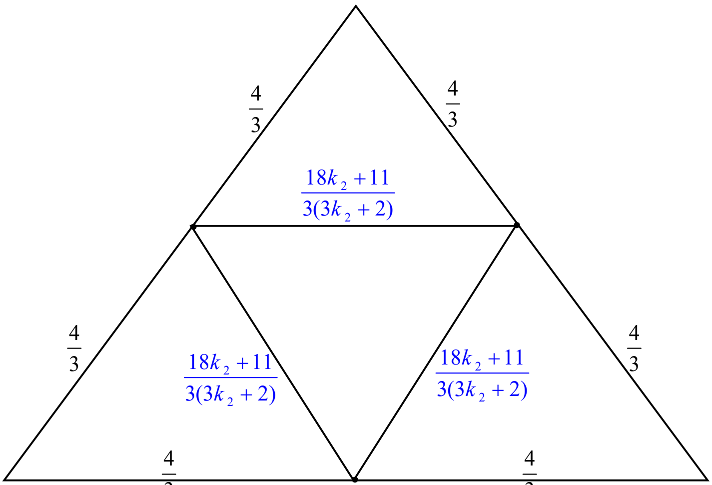
G_6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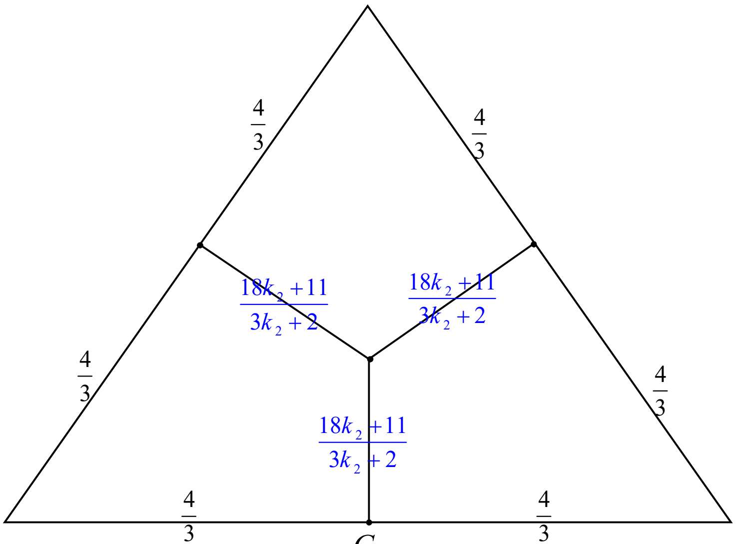
G_7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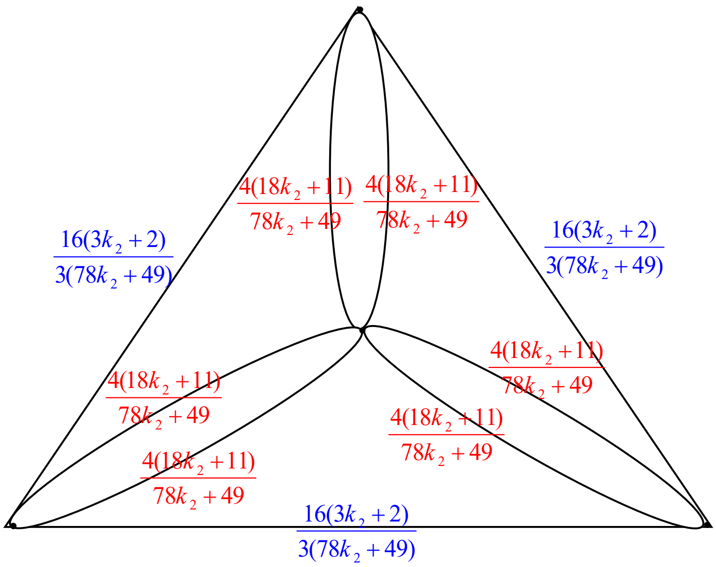
G_8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$\frac{4(18k_2+11)}{78k_2+49}$
$\frac{4(18k_2+11)}{78k_2+49}$
$\frac{4(18k_2+11)}{78k_2+49}$
$\frac{16(3k_2+2)}{3(78k_2+49)}$

G_9

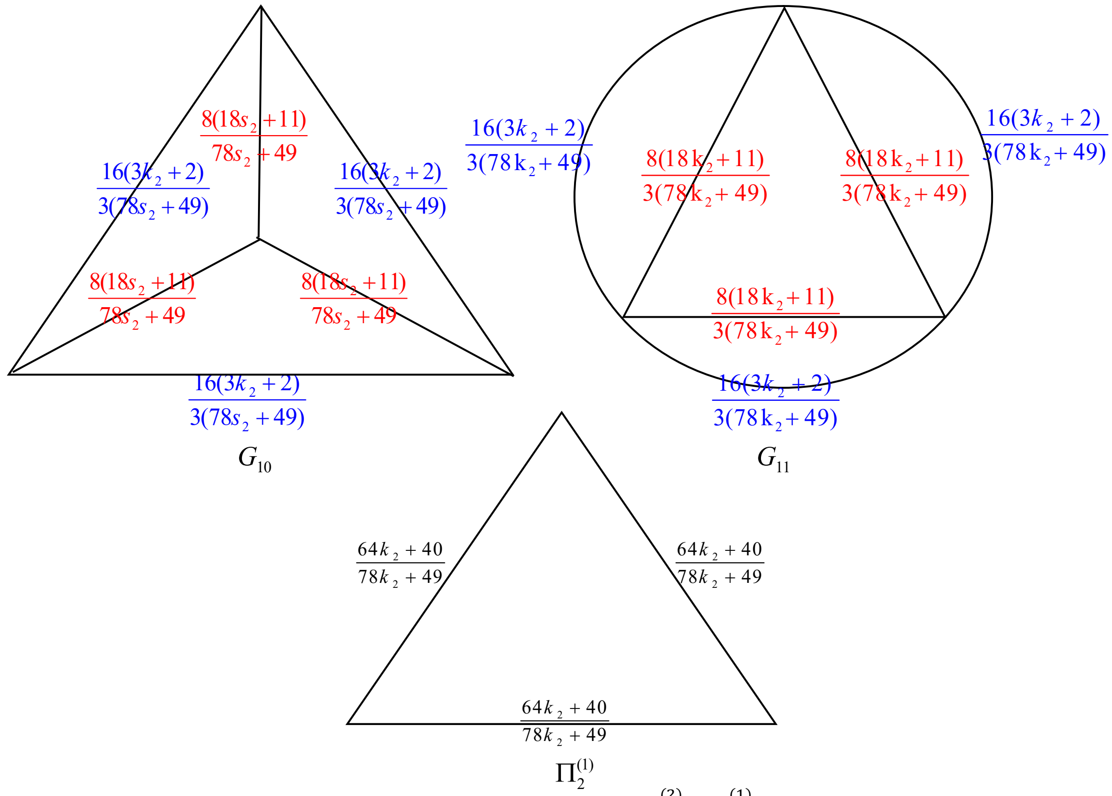


Fig.4 The transformations from $\Pi_2^{(2)}$ to $\Pi_2^{(1)}$

Using the properties given in section 2, we have the following the transformations:
$\tau(G_1)=[\frac{1}{3}]^3\tau(\Pi_2^{(2)}),\tau(G_2)=\tau(G_1),\tau(G_3)=9k_2\tau(G_2),\tau(G_4)=[\frac{1}{3k_2+2}]^3\tau(G_3),\tau(G_5)=\tau(G_4),\tau(G_6)=\left[\frac{3k_2+2}{18k_2}\right]\tau(G_5),\tau(G_7)=\tau(G_6),\tau(G_8)=3[\frac{18k_2+1}{3k_2+2}]\tau(G_7),\tau(G_9)=[\frac{3(3k_2+2)}{78\ \ _2+49}]^3\tau(G_8),\tau(G_{10})=\tau(G_9),\ \tau(G_{11})=[\frac{(78k_2+49)}{24(18k_2+11)^2}]\tau(G_{10})$, and $\tau(\Pi_1^{(1)})=\tau(G_{10})$.

Combining these twelve transformations, we have:

$$\tau(\Pi_2^{(2)})=16(78k_2+49)\tau(\Pi_2^{(1)}). \tag{4.1}$$

Further

$$\tau(\Pi_2^{(n)})=\prod_{i=2}^{n}16(78k_i+49)^2\tau(\Pi_2^{(1)})=3\times16^{n-1}k_1^2[\prod_{i=2}^{n}(78k_i+49)]^2, \tag{4.2}$$

where $k_{i-1}=\frac{64k_i+40}{78k_i+49}, i=2,3,\ldots,n.$

Its characteristic equation is $78\alpha^2-15\alpha-40=0$ with roots $\alpha_1=\frac{15-11\sqrt{105}}{156}$ and $\alpha_2=\frac{15+11\sqrt{105}}{156}$. Subtracting these two roots into both sides of $k_{i-1}=\frac{64\ \ _i+40}{78k_i+49}$, we get

$$k_{i-1}-\frac{15-1\ \sqrt{105}}{156}=\frac{64k_i+4}{78\ \ _i+49}-\frac{15-11\sqrt{105}}{156}=\frac{(113+\ \ \sqrt{105})(k_i-\frac{15-1\ \sqrt{105}}{156})}{2(78k_i+49)} \tag{4.3}$$

$$k_{i-1}-\frac{15+1\ \sqrt{105}}{156}=\frac{64k_i+40}{78k_i+49}-\frac{15+11\sqrt{105}}{156}=\frac{(113-\ \ \sqrt{105})(k_i-\frac{15+1\ \sqrt{105}}{156})}{2(78\ \ _i+4\ \ )} \tag{4.4}$$

Let $h_i=\frac{k_i+\frac{15-1\ \sqrt{105}}{156}}{k_i+\frac{15+11\sqrt{105}}{156}}$.Then by Eqs. (4.3) and (4.4), we get $h_{i-1}=(\frac{12737+124\ \ \sqrt{105}}{32})h_i$ and $h_i=(\frac{12737+1243\sqrt{105}}{32})^{n-i}h_n$.

Therefore, $k_i=\frac{(\frac{12737+1243\sqrt{105}}{32})^{n-i}(\frac{15+1\ \sqrt{105}}{156})h_n-(\frac{15-11\ \sqrt{105}}{156})}{(\frac{12737+1243\sqrt{105}}{32})^{n-i}h_n-1}$.

Thus

$$k_1=\frac{(\frac{12737+124\ \ \sqrt{105}}{32})^{n-1}(\frac{15+11\sqrt{105}}{156})h_n-(\frac{15-113\sqrt{105}}{156})}{(\frac{12737+1243\sqrt{105}}{32})^{n-1}h_n-1}. \tag{4.5}$$

Using the expression $k_{n-1}=\frac{64\ \ _n+40}{78k_n+49}$ and denoting the coefficients of $64k_n+40$ and $78k_n+49$ as f_n and g_n , we have

$$78k_n+49=f_0(64k_n+40)+g_0(78k_n+49),$$

$$78k_{n-1}+49=\frac{f_1(64k_n+40)+g_1(78k_n+49)}{f_0(64k_n+40)+g_0(78k_n+49)},$$
$$78k_{n-2}+49=\frac{f_2(64k_n+40)+g_2(78k_n+49)}{f_1(64k_n+40)+g_1(78k_n+49)},$$
$$\vdots$$
$$78k_{n-i}+49=\frac{f_i(64k_n+40)+g_i(78k_n+4\ )}{f_{i-1}(64k_n+40)+g_{i-1}(78\ \ _n+49)}, \tag{4.6}$$
$$78k_{n-(i+1)}+49=\frac{f_{i+1}(64\ \ _n+40)+g_{i+1}(78\ \ _n+49)}{f_i(64k_n+40)+g_i(78k_n+49)}, \tag{4.7}$$
$$\vdots$$
$$78k_2+49=\frac{f_{n-2}(64k_n+40)+g_{n-2}(78k_n+49)}{f_{n-3}(64k_n+40)+g_{n-3}(78k_n+49)},$$

Thus, we obtain
$$\tau(\Pi_2^{(2)})=3\times 16^{n-1}k_1^2[f_{n-2}(64k_n+40)+g_{n-2}(78k_n+49)]^2 \tag{4.8}$$
where $f_0=0, g_0=1$ and $f_1=78, g_1=49$. By the expression $k_{n-1}=\frac{64k_n+40}{78k_n+49}$ and using Eqs. (4.6) and (4.7), we have
$$f_{i+1}=113f_i-16f_{i-1};\ g_{i+1}=113g_i-16g_{i-1} \tag{4.9}$$
The characteristic equation of Eq. (4.9) is $\beta^2-113\beta+16=0$ with roots $\beta_1=\frac{113+11\sqrt{105}}{2}$ and $\beta_1=\frac{113-\ \ \sqrt{105}}{2}$.
The general solutions of Eq. (4.9) are
$f_i=a_1\beta_1^i+a_2\beta_2^i;\ g_i=b_1\beta_1^i+b_2\beta_2^i$.
Using the initial conditions $a_0=0, b_0=1$ and $a_1=78, b_1=49$, yield
$$f_i=\frac{78\sqrt{105}}{1155}\left(\frac{113+11\sqrt{105}}{2}\right)^i-\frac{78\sqrt{105}}{1155}\left(\frac{113-11\sqrt{105}}{2}\right)^i;$$
$$g_i=\left(\frac{231-3\sqrt{105}}{462}\right)\left(\frac{113+\ \ \sqrt{105}}{2}\right)^i+\left(\frac{231+3\sqrt{105}}{462}\right)\left(\frac{113-11\sqrt{105}}{2}\right)^i \tag{4.10}$$
If $k_n=1$, it means that $\Pi_2^{(n)}$ is without any electrically equivalent transformation. Plugging Eq. (4.10) into Eq. (4.8), we have
$$\tau(\Pi_2^{(n)})=3\times 16^{n-1}k_1^2\left[\left(\frac{97790+9546\sqrt{105}}{1540}\right)\left(\frac{113+\ \ \sqrt{105}}{2}\right)^{n-2}+\left(\frac{97790-9546\sqrt{105}}{1540}\right)\left(\frac{113\ \ \ \ \sqrt{105}}{2}\right)^{n-2}\right]^2, n\geq 2. \tag{4.11}$$
When $n=1$, $\tau(\Pi_2^{(1)})=3$ which satisfies Eq. (4.11). Therefore, the number of spanning trees in the sequence of the graph $\Pi_2^{(n)}$ is given by
$$\tau(\Pi_2^{(n)})=3\times 16^{n-1}k_1^2\left[\left(\frac{97790+9546\sqrt{105}}{1540}\right)\left(\frac{113+\ \ \sqrt{105}}{2}\right)^{n-2}+\left(\frac{97790-9546\sqrt{105}}{1540}\right)\left(\frac{113-\ \ \sqrt{105}}{2}\right)^{n-2}\right]^2, n\geq 1. \tag{4.12}$$
where
$$k_1=\frac{\left(\frac{12737+1243\sqrt{105}}{38}\right)^{n-1}(6525+649\sqrt{105})-\frac{23}{2}(15-1\ \ \sqrt{105})}{\left(\frac{12737+1243\sqrt{105}}{38}\right)^{n-1}(5431+517\sqrt{105})-1794}, n\geq 1. \tag{4.13}$$
Inserting Eq. (4.13) into Eq. (4.12) we obtain the desired result. □

5. Number of spanning trees in the sequences of $\Pi_3^{(n)}$ graph

The graph $\Pi_3^{(n)}$ is defined recursively using the graphs $\Pi_3^{(1)}$ (triangle or K_3) and $\Pi_3^{(2)}$ as shown in Figure 5. The graph $\Pi_3^{(1)}$, n = 3 is obtained by replacing the central triangle in $\Pi_3^{(2)}$ by a copy of $\Pi_3^{(2)}$. In general, $\Pi_3^{(n)}$ is obtained by replacing the central triangle in $\Pi_3^{(n-1)}$ with $\Pi_3^{(2)}$. According to this construction, the number of total vertices $|V(\Pi_3^{(n)})|$ and edges $|E(\Pi_3^{(n)})|$ are $|V(\Pi_3^{(n)})|=9n-6$ and $|E(\Pi_3^{(n)})|=24n-21$, n = 1, 2, ... The average degree of $\Pi_3^{(n)}$ is in the large n limit which is $\frac{16}{3}$.

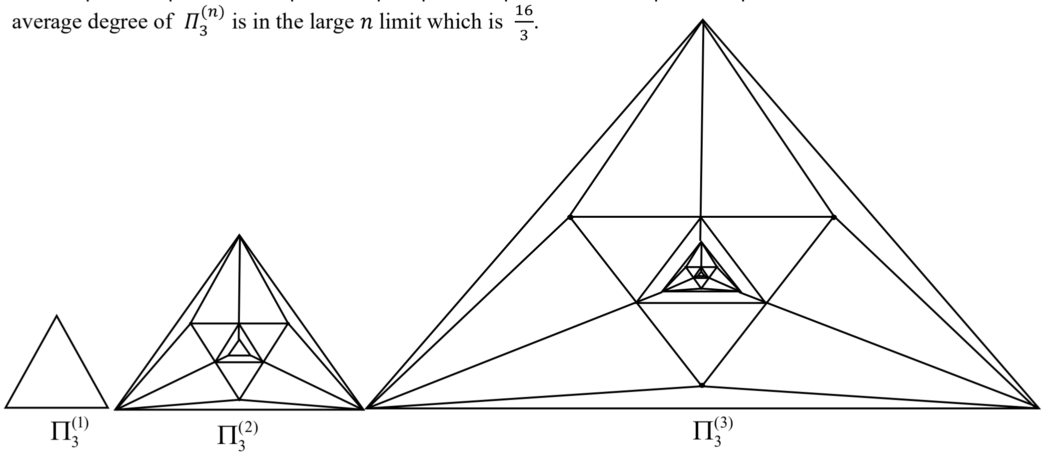
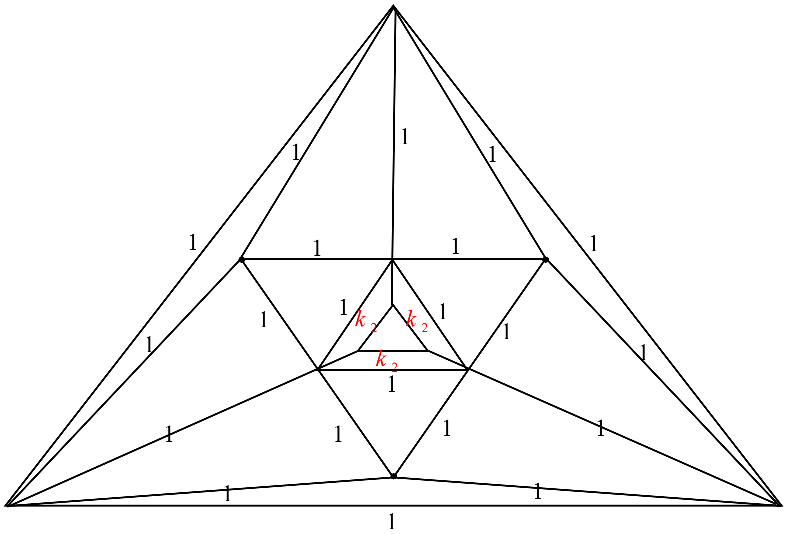


Fig. 5 Some sequences of the graph $\Pi_3^{(n)}$

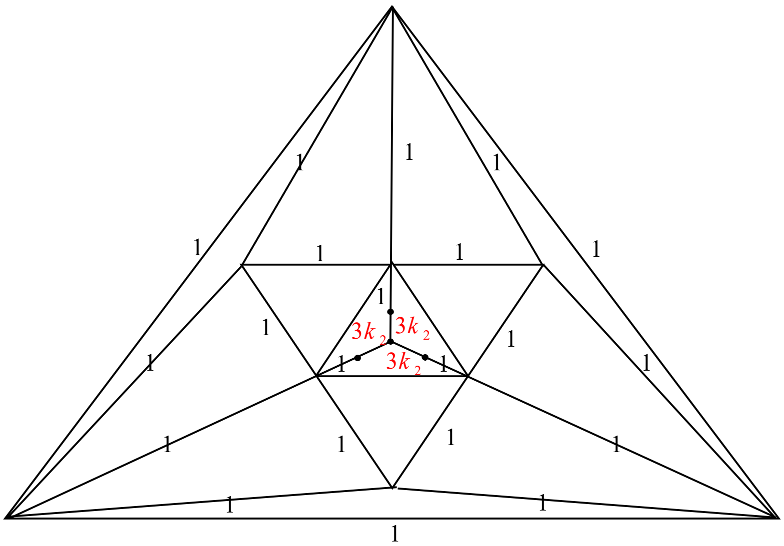
Theorem 3. For $n \geq 1$, the number of spanning trees in the sequence of the graph $\Pi_3^{(n)}$ is given by

$$\frac{(2^{3n-5}((6188-647\sqrt{91})(86+9\sqrt{91})^n+(86-9\sqrt{91})^n(6188+647\sqrt{91}))^2(25(343+41\sqrt{91})-29(\frac{25}{14767+154\ \sqrt{91}})^n(37499+393\ \sqrt{91}))^2)}{(15526875(25(95+4\sqrt{91})+87\times 25^n(14767+1548\sqrt{91})^{1-n})^2)}$$

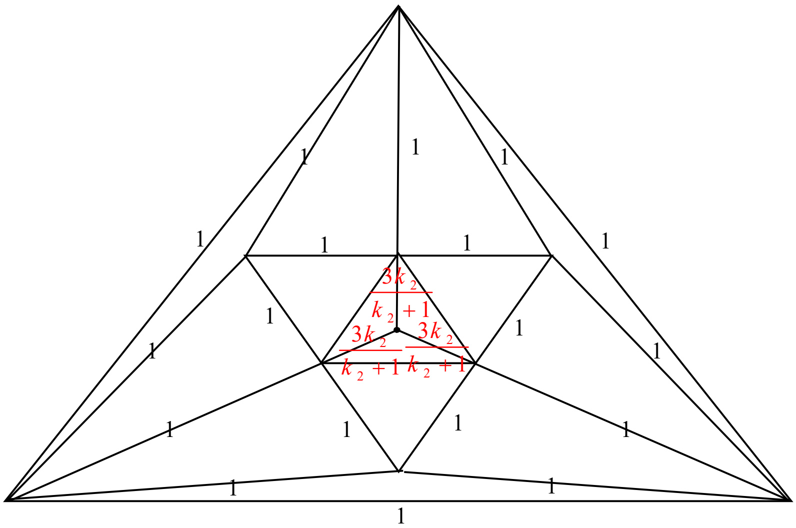
Proof: We use the electrically equivalent transformation to transform $\Pi_3^{(i)}$ to $\Pi_3^{(i-1)}$. Fig.6 illustrates the transformation process from $\Pi_3^{(2)}$ to $\Pi_3^{(1)}$.



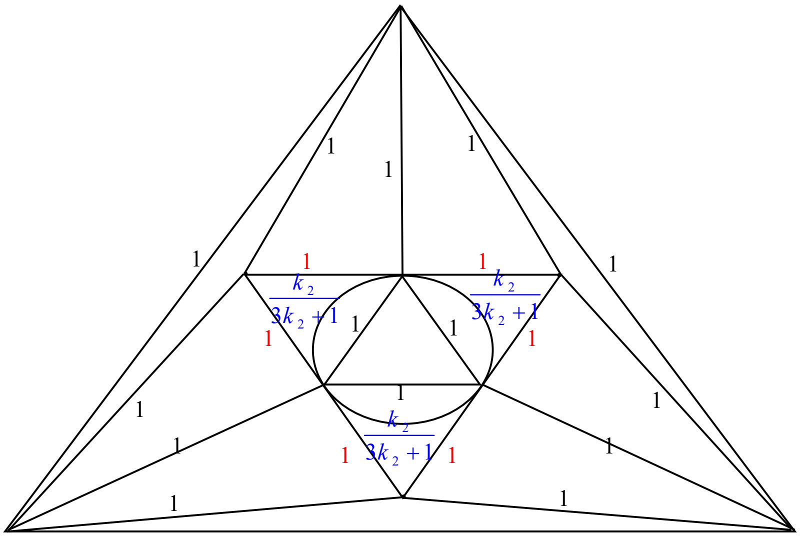
$\Pi_3^{(2)}$



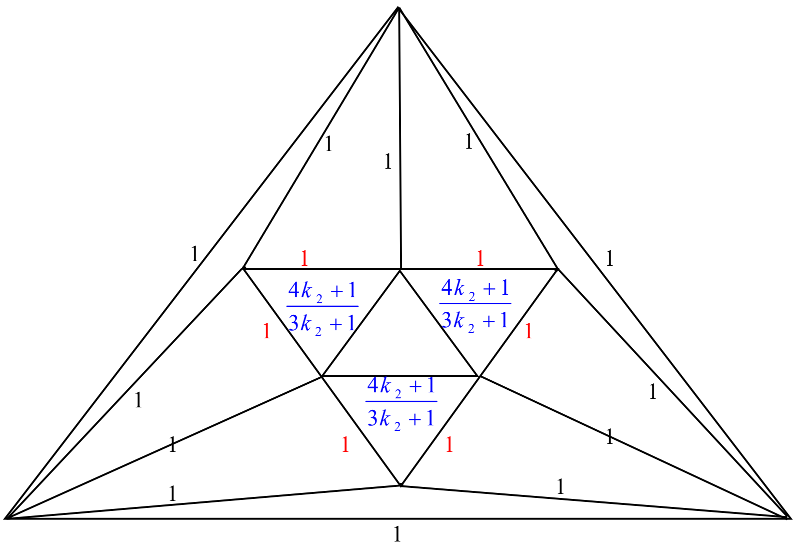
G_1


1
$\frac{3k_2}{k_2+1}$
$\frac{3k_2}{k_2+1}$
$\frac{3k_2}{k_2+1}$

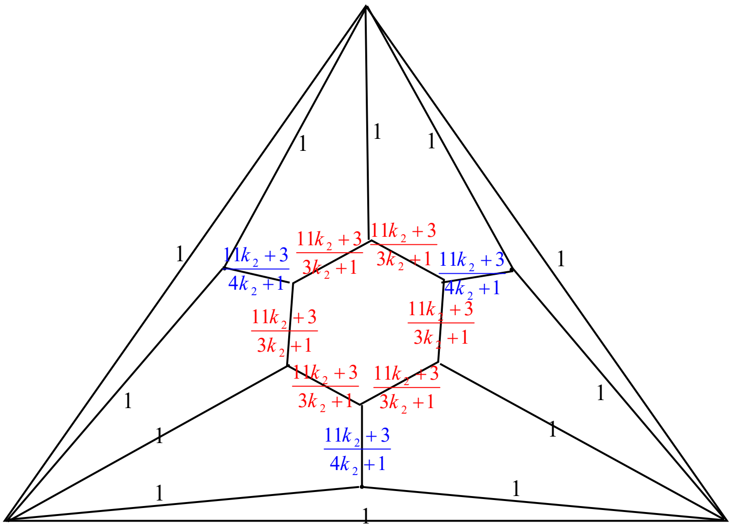
G_2


1
$\frac{k_2}{3k_2+1}$
$\frac{k_2}{3k_2+1}$
$\frac{k_2}{3k_2+1}$

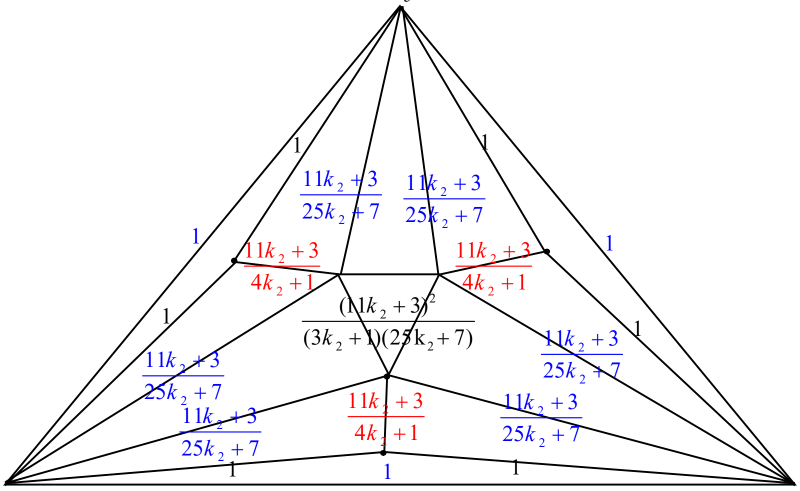
G_3


1
$\frac{4k_2+1}{3k_2+1}$
$\frac{4k_2+1}{3k_2+1}$
$\frac{4k_2+1}{3k_2+1}$

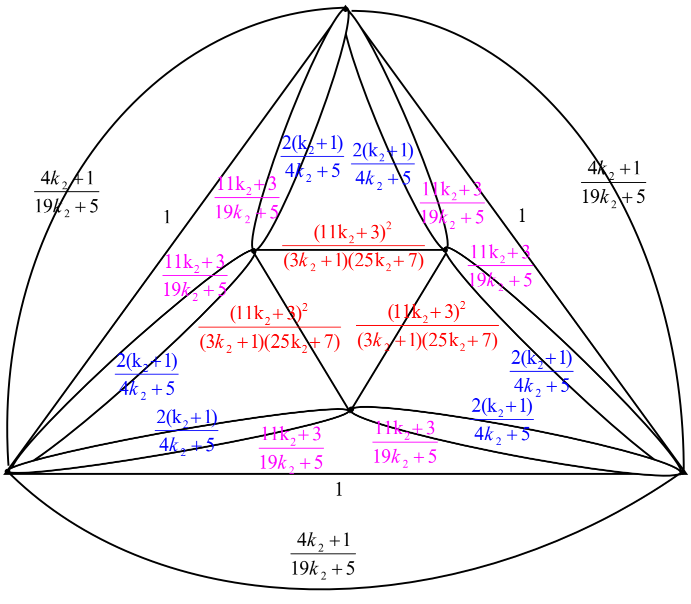
G_4


1
1
1
1
1
1
1
1
1
1
1
1
1
$\frac{11k_2+3}{4k_2+1}$
$\frac{11k_2+3}{4k_2+1}$
$\frac{11k_2+3}{4k_2+1}$
$\frac{11k_2+3}{3k_2+1}$
$\frac{11k_2+3}{3k_2+1}$
$\frac{11k_2+3}{3k_2+1}$
$\frac{11k_2+3}{3k_2+1}$
$\frac{11k_2+3}{3k_2+1}$
$\frac{11k_2+3}{3k_2+1}$

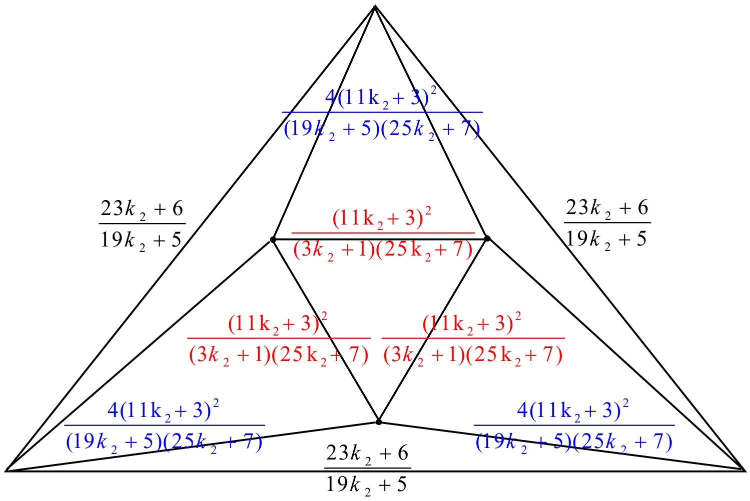
G_5


1
1
1
1
1
1
1
1
1
$\frac{11k_2+3}{25k_2+7}$
$\frac{11k_2+3}{25k_2+7}$
$\frac{11k_2+3}{25k_2+7}$
$\frac{11k_2+3}{25k_2+7}$
$\frac{11k_2+3}{25k_2+7}$
$\frac{11k_2+3}{25k_2+7}$
$\frac{11k_2+3}{4k_2+1}$
$\frac{11k_2+3}{4k_2+1}$
$\frac{11k_2+3}{4k_2+1}$
$\frac{(11k_2+3)^2}{(3k_2+1)(25k_2+7)}$

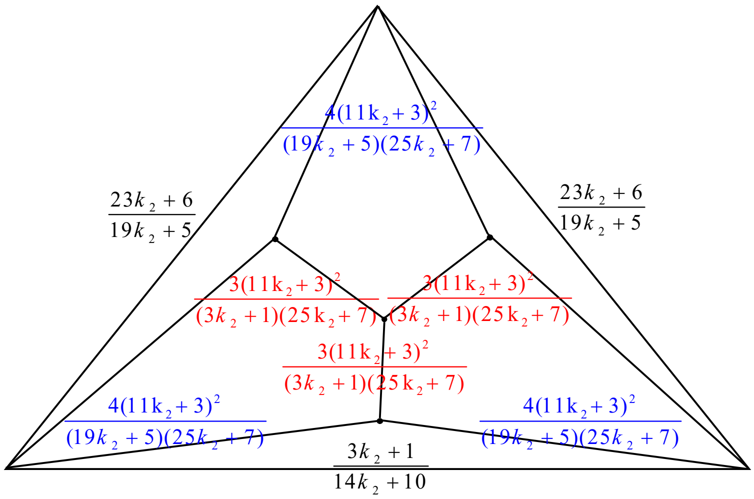
G_6


$\frac{4k_2+1}{19k_2+5}$
$\frac{4k_2+1}{19k_2+5}$
$\frac{4k_2+1}{19k_2+5}$
1
1
1
$\frac{2(k_2+1)}{4k_2+5}$
$\frac{2(k_2+1)}{4k_2+5}$
$\frac{2(k_2+1)}{4k_2+5}$
$\frac{2(k_2+1)}{4k_2+5}$
$\frac{2(k_2+1)}{4k_2+5}$
$\frac{2(k_2+1)}{4k_2+5}$
$\frac{11k_2+3}{19k_2+5}$
$\frac{11k_2+3}{19k_2+5}$
$\frac{11k_2+3}{19k_2+5}$
$\frac{11k_2+3}{19k_2+5}$
$\frac{11k_2+3}{19k_2+5}$
$\frac{11k_2+3}{19k_2+5}$
$\frac{(11k_2+3)^2}{(3k_2+1)(25k_2+7)}$
$\frac{(11k_2+3)^2}{(3k_2+1)(25k_2+7)}$
$\frac{(11k_2+3)^2}{(3k_2+1)(25k_2+7)}$

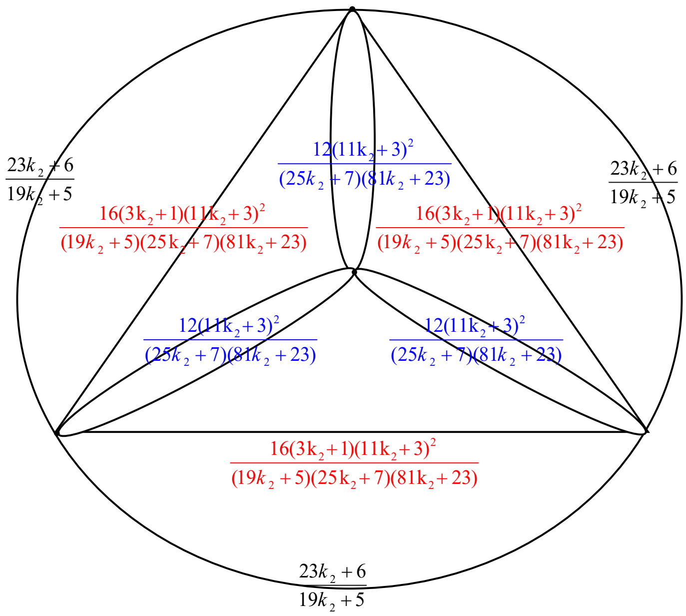
G_7


$\frac{4(11k_2+3)^2}{(19k_2+5)(25k_2+7)}$
$\frac{23k_2+6}{19k_2+5}$
$\frac{(11k_2+3)^2}{(3k_2+1)(25k_2+7)}$
$\frac{23k_2+6}{19k_2+5}$
$\frac{(11k_2+3)^2}{(3k_2+1)(25k_2+7)}$
$\frac{(11k_2+3)^2}{(3k_2+1)(25k_2+7)}$
$\frac{4(11k_2+3)^2}{(19k_2+5)(25k_2+7)}$
$\frac{4(11k_2+3)^2}{(19k_2+5)(25k_2+7)}$
$\frac{23k_2+6}{19k_2+5}$

G_8


$\frac{4(11k_2+3)^2}{(19k_2+5)(25k_2+7)}$
$\frac{23k_2+6}{19k_2+5}$
$\frac{23k_2+6}{19k_2+5}$
$\frac{3(11k_2+3)^2}{(3k_2+1)(25k_2+7)}$
$\frac{3(11k_2+3)^2}{(3k_2+1)(25k_2+7)}$
$\frac{3(11k_2+3)^2}{(3k_2+1)(25k_2+7)}$
$\frac{4(11k_2+3)^2}{(19k_2+5)(25k_2+7)}$
$\frac{4(11k_2+3)^2}{(19k_2+5)(25k_2+7)}$
$\frac{3k_2+1}{14k_2+10}$

G_9


$\frac{23k_2+6}{19k_2+5}$
$\frac{12(11k_2+3)^2}{(25k_2+7)(81k_2+23)}$
$\frac{23k_2+6}{19k_2+5}$
$\frac{16(3k_2+1)(11k_2+3)^2}{(19k_2+5)(25k_2+7)(81k_2+23)}$
$\frac{16(3k_2+1)(11k_2+3)^2}{(19k_2+5)(25k_2+7)(81k_2+23)}$
$\frac{12(11k_2+3)^2}{(25k_2+7)(81k_2+23)}$
$\frac{12(11k_2+3)^2}{(25k_2+7)(81k_2+23)}$
$\frac{16(3k_2+1)(11k_2+3)^2}{(19k_2+5)(25k_2+7)(81k_2+23)}$
$\frac{23k_2+6}{19k_2+5}$

G_{10}

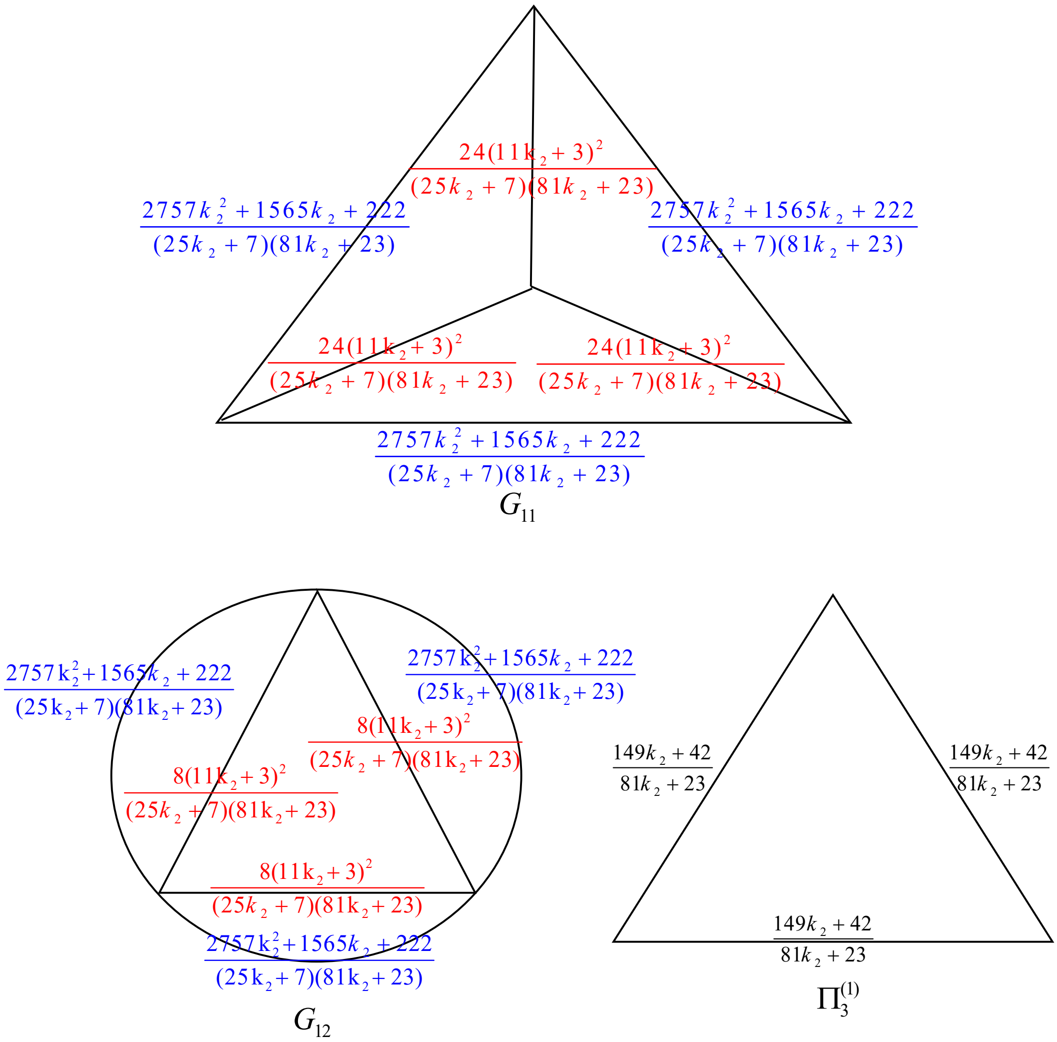


Fig.6 The transformations from $\Pi_3^{(2)}$ to $\Pi_3^{(1)}$

Using the properties given in section 2, we have the following the transformations:
$\tau(G_1) = 9k_2\,\tau(\Pi_2^{(2)}), \tau(G_2) = [\frac{1}{3k_2+1}]^3\tau(G_1), \tau(G_3) = [\frac{3k_2+1}{9k_2}]\tau(G_2), \tau(G_4) = \tau(G_3), \tau(G_5) = [\frac{(11\ {}_2+3)^2}{(3k_2+1)(4k_2+1)}]^3\tau(G_4), \tau(G_6) = [\frac{3k_2+1}{25k_2+7}]^3\tau(G_5), \tau(G_7) = [\frac{4k_2+1}{19k_2+5}]^3\tau(G_6), \tau(G_8) = \tau(G_7), \tau(G_9) = 9[\frac{(11\,k_2+3)^2}{(3k_2+1)(25k_2+7)}]\tau(G_8), \tau(G_{10}) = [\frac{(3k_2+1)(19\ {}_2+5)(25\ {}_2+7)}{(11\,k_2+3)^2(81k_2+23)}]^3\tau(G_9),\ \tau(G_{11}) = \tau(G_{10}), \tau(G_{12}) = \frac{(81k_2+23)(25\ {}_2+7)}{72(11\ {}_2+3)^2}\tau(G_{11})$ and $\tau(\Pi_3^{(1)}) = \tau(G_{12})$.

Combining these thirteen transformations, we have:

$$\tau(\Pi_3^{(2)}) = 8(81k_2+23)\tau(\Pi_3^{(1)}). \tag{5.1}$$

Further

$$\tau(\Pi_3^{(n)}) = \prod_{i=2}^{n} 8(81k_i+23)^2\tau(\Pi_3^{(1)}) = 3\times 8^{n-1}k_1^2[\prod_{i=2}^{n}(81k_i+23)]^2, \tag{5.2}$$

where $k_{i-1} = \frac{149k_i+42}{81k_i+23}, i = 2,3,\dots,n.$

Its characteristic equation is $81\alpha^2 - 126\alpha - 42 = 0$ with roots $\alpha_1 = \frac{7-\sqrt{91}}{9}$ and $\alpha_2 = \frac{7+\sqrt{91}}{9}$. Subtracting these two roots into both sides of $k_{i-1} = \frac{149k_i+42}{81\ {}_i+2}$, we get

$$k_{i-1} - \frac{7-\sqrt{91}}{9} = \frac{149k_i+42}{81k_i+23} - \frac{7-\sqrt{91}}{9} = \frac{(86+9\sqrt{91})(k_i-\frac{7-\sqrt{91}}{9})}{81(k_i+23)} \tag{5.3}$$

$$k_{i-1} - \frac{7+\sqrt{91}}{9} = \frac{149k_i+4}{81k_i+23} - \frac{7+\sqrt{91}}{9} = \frac{(86-9\sqrt{91})(k_i-\frac{7+\sqrt{91}}{9})}{81(k_i+2\ \ )} \tag{5.4}$$

Let $h_i = \frac{k_i - \frac{7-\sqrt{91}}{9}}{k_i + \frac{7+\sqrt{91}}{9}}$. Then by Eqs. (5.3) and (5.4), we get $h_{i-1} = (\frac{14767+154\ \sqrt{91}}{25})h_i$ and $h_i = (\frac{14767+1548\sqrt{91}}{25})^{n-i}h_n$.

Therefore $k_i = \frac{(\frac{14767+154\ \sqrt{91}}{25})^{n-i}(\frac{7+\sqrt{91}}{9})h_n - (\frac{7-\sqrt{91}}{9})}{(\frac{14767+1548\sqrt{91}}{25})^{n-i}h_n - 1}$.

Thus

$$k_1 = \frac{(\frac{14767+1548\sqrt{91}}{25})^{n-i}(\frac{7+\sqrt{91}}{9})h_n - (\frac{7-\sqrt{91}}{9})}{(\frac{14767+1548\sqrt{91}}{25})^{n-i}h_n - 1}. \quad (5.5)$$

Using the expression $k_{n-1} = \frac{149k_n+42}{81k_n+2}$ and denoting the coefficients of $149k_n + 42$ and $81k_n + 23$ as f_n and g_n , we have

$$81k_n + 23 = f_0(149k_n + 42) + g_0(81k_n + 23),$$
$$81k_{n-1} + 23 = \frac{f_1(149k_n + 42) + g_1(81k_n + 23)}{f_0(149k_n + 42) + g_0(81k_n + 23)},$$
$$81k_{n-2} + 23 = \frac{f_2(149k_n + 42) + g_2(81k_n + 23)}{f_1(149k_n + 42) + g_1(81k_n + 23)},$$
$$\vdots$$
$$81k_{n-i} + 23 = \frac{f_i(149k_n+42)+g_i(81k_n+23)}{f_{i-1}(149k_n+42)+g_{i-1}(81k_n+2\ )}, \quad (5.6)$$
$$81k_{n-(i+1)} + 23 = \frac{f_{i+1}(149k_n+42)+g_{i+1}(81\ _n+23)}{f_i(149k_n+42)+g_i(81\ _n+23)}, \quad (5.7)$$
$$\vdots$$
$$81k_2 + 23 = \frac{f_{n-2}(149k_n + 42) + g_{n-2}(81k_n + 23)}{f_{n-3}(149k_n + 42) + g_{n-3}(81k_n + 23)},$$

Thus, we obtain

$$\tau(\Pi_3^{(n)}) = 3 \times 8^{n-1}k_1^2[f_{n-2}(149k_n + 42) + g_{n-2}(81k_n + 23)]^2 \quad (5.8)$$

where $f_0 = 0, g_0 = 1$ and $f_1 = 81, g_1 = 23$. By the expression $k_{n-1} = \frac{149k_n+42}{81k_n+23}$ and using Eqs. (5.6) and (5.7), we have $f_{i+1} = 172f_i - 25f_{i-1}; g_{i+1} = 172g_i - 25g_{i-1}$ (5.9)

The characteristic equation of Eq. (5.9) is $\beta^2 - 172\beta + 25 = 0$ with roots $\beta_1 = 86 + 9\sqrt{91}$ and $\beta_1 = 86 + 9\sqrt{91}$.
The general solutions of Eq. (5.9) are

$f_i = a_1\beta_1^i + a_2\beta_2^i; g_i = b_1\beta_1^i + b_2\beta_2^i$.

Using the initial conditions $a_0 = 0, b_0 = 1$ and $a_1 = 81, b_1 = 23$, yields

$$f_i = \frac{9\sqrt{91}}{182}(86 + 9\sqrt{91})^i - \frac{9\sqrt{91}}{182}(86 - 9\sqrt{91})^i; g_i = (\frac{91-7\sqrt{91}}{182})(86 + 9\sqrt{91})^i + (\frac{91+7\sqrt{91}}{182})(86 - 9\sqrt{91})^i \quad (5.10)$$

If $k_n = 1$, it means that $\Pi_3^{(n)}$ is without any electrically equivalent transformation. Plugging Eq. (5.10) into Eq. (5.8), we have

$$\tau(\Pi_3^{(n)}) = 3 \times 8^{n-1}k_1^2[(\frac{9464+991\sqrt{91}}{182})(86 + 9\sqrt{91})^{n-2} + (\frac{9464-991\sqrt{91}}{182})(86 - 9\sqrt{91})^{n-2}]^2, n \geq 2. \quad (5.11)$$

When $n=1$, $\tau(\Pi_3^{(1)}) = 3$ which satisfies Eq. (5.11). Therefore, the number of spanning trees in the sequence of the graph $\Pi_3^{(n)}$ is given by

$$\tau(\Pi_3^{(n)}) = 3 \times 8^{n-1}k_1^2[(\frac{9464+991\sqrt{91}}{182})(86 + 9\sqrt{91})^{n-2} + (\frac{9464-991\sqrt{91}}{182})(86 - 9\sqrt{91})^{n-2}]^2, n \geq 1. \quad (5.12)$$

where

$$k_1 = \frac{(\frac{14767+1548\sqrt{91}}{25})^{n-1}(343+4\ \sqrt{91})+29(7-\sqrt{91})}{3(\frac{14767+1548\sqrt{91}}{25})^{n-1}(95+4\sqrt{91})+261}, n \geq 1. \quad (5.13)$$

Inserting Eq. (5.13) into Eq. (5.12) we obtain the desired result. □

6. Numerical Results

The values of the number of spanning trees in the graphs $\tau(\Pi_1^{(n)}), \tau(\Pi_2^{(n)})$ and $\tau(\Pi_3^{(n)})$ are shown in the following three tables:

Table (1) shows some values of the numbers for the graph $\tau(\Pi_1^{(n)}), \tau(\Pi_2^{(n)})$ and $\tau(\Pi_3^{(n)})$.

n	$\tau(\Pi_1^{(n)})$	$\tau(\Pi_2^{(n)})$	$\tau(\Pi_3^{(n)})$
1	3	3	3
2	681210	519168	875544
3	185510306700	105779478528	206901490368
4	50862671085249000	21556971675844608	48885000926148096
5	13948735120763776830000	4393131437918055825408	11550148664166384218112
6	382537659619528346445210	8952836292381213531077345	27289747697801635132579184

7. Spanning Tree Entropy

After having explicit Formulas for the number of spanning trees of the sequence of the six graphs $\Pi_1^{(n)}, \Pi_2^{(n)}$ and $\Pi_3^{(n)}$, we can calculate its spanning tree entropy Z which is a finite number and a very interesting quantity characterizing the network structure, defined as in [21,22] as: For a graph G,

$$Z(G) = \lim_{n\to\infty} \frac{\ln\ (G)}{|V(G)|}. \qquad (7.1)$$

$$Z(\Pi_1^{(n)}) = \frac{1}{9}\left(\ln\left[\frac{1215}{2}\right] + 2\ln\left[11+\sqrt{105}\right]\right) = 1.391.$$

$$Z(\Pi_2^{(n)}) = \frac{1}{9}(\ln[4] + 2\ln[113 + 11\sqrt{105}]) = 1.358.$$

$$Z(\Pi_3^{(n)}) = \frac{1}{9}(\ln[8] + 2\ln[86 + 9\sqrt{91}]) = 1.375.$$

Now we compare the value of entropy in our graphs with other graphs. It is clear that the entropy of the graph $\Pi_1^{(n)}$ is larger than the entropy of the graphs $\Pi_2^{(n)}$and $\Pi_3^{(n)}$ of the same average degree. In addition the entropies of the graphs $\Pi_1^{(n)}, \Pi_2^{(n)}$ and $\Pi_3^{(n)}$ are larger than the entropy of the fractal scale free lattice [23] which has the entropy1.040 and has the same average degree 4 and the entropy of the two dimensional Sierpinski gasket [24] which has the entropy1.166 of the same average degree 4 . Moreover, the entropies of the graphs $\Pi_1^{(n)}, \Pi_2^{(n)}$ and $\Pi_3^{(n)}$ are larger than entropy of the 3-prism graph [25] which has the entropy1.0445.

8. Conclusions

In this study, we use electrically equivalent transformations to determine the number of spanning trees in the sequences of various graphs created by triangle graphs. This technique's strength is its ability to avoid the laborious computation of Laplacian spectra, which is a requirement for a general approach to spanning tree determination. Furthermore, our findings indicate a relationship between the entropy and the graph's average degree.

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