A Brief Evaluation of Level Based Entropy Measures and A New Information Entropy

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Abstract. In this paper, we defined a new graph entropy which is called level entropy. There are some level based entropy measures in the literature. Then, we obtain some relations between the level entropy and other level based entropy measures for rooted trees. Moreover, we obtain that level entropy is less than or equal to Wiener entropy of a rooted tree, if the root vertex of this tree is one of the most distant vertices to any vertex.

Keywords: Network, Graph Entropy, Shortest Path, Level, Height

1. Introduction

Information theoretic tools or entropy measures are widely used in mathematics, computer sciences, information sciences, chemistry, biology, sociology and psychology (Dehmer and Mowshowitz 2011). Then, the new methods of measuring the entropy of graph classes will easify the researches in the mentioned areas. The information contents of graphs can be evaluated as a complexity measure for the graphs (Mowshowitz and Dehmer 2012).

To enumerate the entropy measure of a graph, a probability distribution has to be obtained with respect to a parameter. A way of the obtaining the probability distribution is using an equivalence relation on a graph (Rashevsky 1955, Trucco 1956). One of the first studies on these directions contain the symmetry properties and permutation groups of graphs (Mowshowitz 1968). Using the algebraic methods on the graphs give different methods for computing the information contents of the graphs. By these methods, different partitions of the the vertex set or edge set can be obtained and these partitions attain the probability distribution on graphs. Orbit polynomial (Dehmer et al. 2020) and orbit entropy concepts are some of these studies.

Distance based entropies have important role in information theory. There are many distance based entropy measures in the literature. The first one is called Wiener entropy (Cambie and Dong 2025) or distance entropy (Ghorbani et al. 2019) which is computed by the distance of a vertex to other vertices. Some of the distance based entropy measures can be ordered as (Ilić and Dehmer 2015, Chen et al. 2014).

Some entropy measures defined by different graph parameters which are based vertex degrees (Cao et al. 2015), number of matchings and independent sets (Cao et al. 2017, Wan et al. 2020) and number of dominating sets (Şahin 2022).

Level based analysis of complex networks is widely used in information sciences and statistics Level based entropy measures can be obtained from metrical graph properties (Dehmer 2008a). The sphere concept is very useful in calculation of the metrical properties of the vertices. The set of vertices which have the equal distance to a vertex is called sphere of the related vertex.

Another way of the level based analaysis is using local property measures (Dehmer 2008a) for example closeness or degree centrality (Brandes 2001, Wasserman and Faust 1994). This entropy measure uses the local information graphs which are well known in the theory of social networks (Brandes 2001, Wasserman and Faust 1994).

A different level based partition can be obtained from the tree decomposition of the graphs. In this method, a probability distribution is obtained for each vertex of the graphs by rooting the graphs at a vertex and counting the the number of vertices of each level. The total decomposition entropy equals to the sum of the probability distribution of each vertex (Dehmer 2008b, Mowshowitz and Dehmer 2012).

It is understood that level based analysis of graphs has also an important role in statistics. Average level or average depth has many applications in statistics (Desmarais and Mahmoud 2020, Bhutani et al. 2020). Since the caterpillar graphs some biological applications, level based analysis of caterpillar graphs especially obtained by Balaji and Mahmoud (2017). The authors defined a distance based graph invariant called level index which is computed by the level differences between the vertices.

Since the level based analysis of graphs have many applications in computer sciences, information sciences, mathematics and statistics, we decided to define a new entropy measure **level entropy** which is based on the level index concept. In this paper, we give an evaluation of the level based entropy measures of the literature and give some relations between these entropies and level entropy.

2. Preliminaries

We give the essential definitions and theorems about the graph theory and level index which are important for this paper (Dehmer and Mowshowitz 2011).

Definition 2.1. A connected and undirected graph G is consisted of the edge set E and the vertex set V.

Definition 2.2. A tree T is acyclic and undirected graph. If a tree rooted T at a vertex u_0, then T is called a rooted tree and u_0 is called the root vertex of T. The level of a vertex u equals to distance of the path between u and u_0. The maximum distance of the path between the root u_0 and any vertex of T is called the height of T. A vertex whose degree one is called a leaf.

Definition 2.3. The length of the shortest path between the vertices u and v is denoted by $d(u, v)$. The eccentricity of a vertex v is denoted by $\sigma(v)$ and equals to maximum distance between v and any vertex of G. The diameter of G is denoted by $p(G) = max_{v \in V} \sigma(v)$ and radius of G is denoted by $r(G) = min_{v \in V} \sigma(v)$

Definition 2.4. The j-sphere of a vertex u_i in G is introduced by

$$S_j(u_i, G) = \{v \in V(G): d(u_i, v) = j, j \geq 1\}.$$

Definition 2.5. The total distance is called Wiener index in a graph G and it eqauals to

$$W(G) = \frac{1}{2} \sum_{u \in V(G)} D(u)$$

where $D(u)$ is used to denote the total distance of $u \in V(G)$ to other vertices of G (Wiener 1947).

Definition 2.6. For a rooted tree T, the difference between the levels of the vertices u and v is denoted by $\ell(u, v)$.

Definition 2.7. Balaji and Mahmoud introduced a distance based topological index for rooted trees (2017). It is called level index and the level index of a tree T is denoted by $L(T)$. Level index of a tree T is computed by the following equation

$$L(T) = \sum_{1 \leq i < j \leq n} |l_j(T) - l_i(T)|$$

such that $l_i(T)$ and $l_j(T)$ showing the distances of the vertices v_i and v_j from the root vertex of the tree T.

In the following example, level index of a rooted tree is given

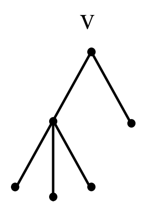

Figure 1. The tree T which is used in the following example

Example 2.8. The level index of T is computed by

$$L(T) = 1 + 1 + 2 + 2 + 2 + 1 + 1 + 1 + 1 + 1 + 1 = 14.$$

Definition 2.9. The sum of differences of levels between a vertex $u \in V(T)$ and the other vertices of T is denoted by $L(u)$ and it is computed by

$$L(u) = \sum_{v \in V(T)} \ell(u, v).$$

It means that the sum of differences of levels between a vertex i and the other vertices equals to the following expression in terms of levels of the vertices. Then $L(i)$ equals to sum of the differences between the level of any vertex j and the vertex i

$$L(i) = \sum_{1 \le j \le n} \left| l_j(T) - l_i(T) \right|.$$

By definition 2.9 we can say that

$$L(T) = \frac{1}{2} \sum_{u \in V(T)} L(u).$$

Balaji and Mahmoud (2017) itroduced a caterpillar from a rooted path $u_0 u_1 \dots u_{h-1}$ with the root u_0 and height h. The path P_h is called spine. Assume that X_i nodes for $i = 1, \dots, h$ are attached to u_{i-1} as a leaf. Therefore for $i = 1, \dots, h-1$, level i has $N_i = X_i + 1$ nodes (the 1 comes from the vertex from the spine on level i), but level h has X_h nodes only.

Definition 2.10. Level index of any rooted tree T can be computed by the following equation with height h and the number of vertices N_i and N_{i+j} at level i and $i + j$ respectively (Balaji and Mahmoud 2017).

$$L(T) = \sum_{i=0}^{h} \sum_{j=0}^{h-j} j N_i N_{i+j}.$$

3. Entropy Measures

We give the definition of entropy measures which are especially based on the levels of the vertices. We define the level entropy by a new information functional.

Definition 3.1. The entropy of a graph G is defined by Dehmer's information functional method (Dehmer 2008a) where an arbitrary information functional denoted by f as follows

$$I_f(G) = -\sum_{i=1}^{|V|} \frac{f(v_i)}{\sum_{j=1}^{|V|} f(v_j)} \log\left(\frac{f(v_i)}{\sum_{j=1}^{|V|} f(v_j)}\right)$$

$$= \log\left(\sum_{i=1}^{[V]} f(v_i)\right) - \sum_{i=1}^{|V|} \frac{f(v_i)}{\sum_{j=1}^{[V]} f(v_j)} \log f(v_i).$$

We note that logarithms are taken to the base 2.

By Definition 3.1, $f(v_i)$ can be choosen with respect to different graph parameters and different entropy measures can be obtained.

The Wiener entropy is given by the following definition.

Definition 3.2. The Wiener entropy a graph G is denoted by I_w and it is computed by (Cambie and Dong 2025)

$$I_w(G) = -\sum_{u\in V(G)} \frac{D(u)}{2W(G)} \log \frac{D(u)}{2W(G)}$$

$$= 1 + 2\log W(G) - \frac{1}{2W(G)} \sum_{u\in V(G)} D(u) \log D(u).$$

The decomposition of graphs is a very useful method in mathematics and computer science. In this method, the graphs are decomposed to smaller graphs with a special property. We use the decomposition of graphs which is based on vertex partitions with respect to levels of vertices (Dehmer 2008b). Determining of vertex partitions is computed as follows. The graph is rooted to a vertex and the the other vertices ordered with respect to their levels. The number of vertices on each level gives a partition of the vertices. This rule is repeated for each vertex of the graph.

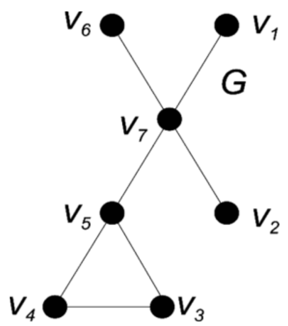

Figure 2. The graph G which is used for decomposition entropy

The famous Dijkstra Alghoritm (1959) can be used in determining the decomposition of a graph G into set of generalized trees (Dehmer 2008b, Dehmer and Mowshowitz 2011). The decomposition of the graph G in Figure 2 is depicted in Figure 3 (Dehmer 2008b). In Figure 3, it is seen that the graph G is decomposing into a set of generalized trees which are rooted tree-like structures.

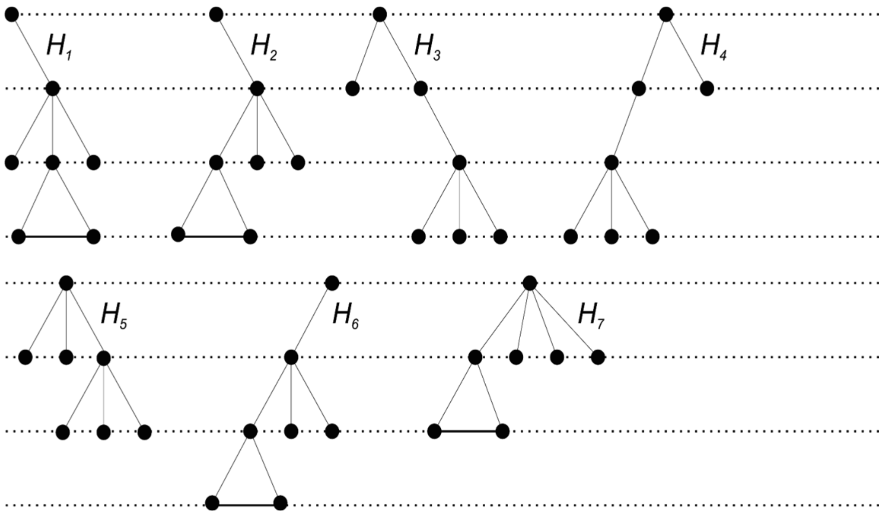

Figure 3. The decomposition of the graph G to set of generalized trees for each vertex

Definition 3.3. Let H be a rooted tree with height h and order n. Moreover, n_i denotes the number of vertices on the i-th level. A probability distribution to H is defined by (Emmert-Streib and Dehmer 2007)

$$p_i^V := \frac{n_i}{n}$$

By this way the vertex entropy of H is given by

$$I^V(H) = -\sum_{i=1}^{h} p_i^V \log p_i^V.$$

The total decomposition entropy of a rooted tree is given as follows. Let T be a rooted tree and $S_H^T = \{H_1, H_2, \ldots, H_n\}$ be the associated set of trees as exemplified in Figure 3. Then the total decomposition entropy is defined by

$$I^V(T) = \sum_{i=1}^{n} I^V(H_i).$$

Now we can give two information functionals which are based on metrical properties of graphs. In these definitions the sphere concept has an important role in the determining the probability distributions.

Definition 3.4. Let G be a graph. For a vertex $v_i \in V$ with the height p, the information functional can be given by (Dehmer 2008b).

$$f^V(v_i) = \alpha^{c_1|S_1(v_i,G)|+c_2|S_2(v_i,G)|+\cdots+c_p|S_p(v_i,G)|}$$

with $c_k > 0, 1 \leq k \leq p, \alpha > 0$. In here c_k are arbitrary real positive numbers. $S_j(v_i, G)$ is used to show the j-sphere of v_i in G and $|S_j(v_i, G)|$ its cardinality.

Definition 3.5. By the definition of information functional of Definition 3.4, we can give the following entropy

$$I_{f^V}(G) = -\sum_{i=1}^{|V|} \frac{f^V(v_i)}{\sum_{j=1}^{|V|} f^V(v_i)} \log\left(\frac{f^V(v_i)}{\sum_{j=1}^{|V|} f^V(v_i)}\right).$$

A different metrical property based information functional f^P is defined by the path lengths of the local information graph $L_G(v_i, j)$ for each vertex v_i and $j = 1,2, \ldots, p$.

The Figure 4 illustrates the j-spheres of a graph G by centric circles for $j = 1,2,3$ and Figure 5 illustrates the local information graphs $L_G(v_i, 1)$ $L_G(v_i, 2)$ and $L_G(v_i, 3)$ (Dehmer and Mowshowitz 2011).

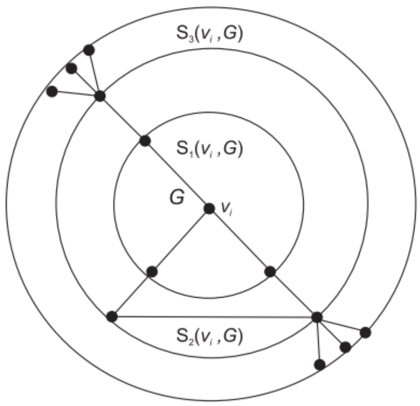

Figure 4. The centric circles show the 1-sphere, 2-sphere and 3-sphere of v_i, resp.

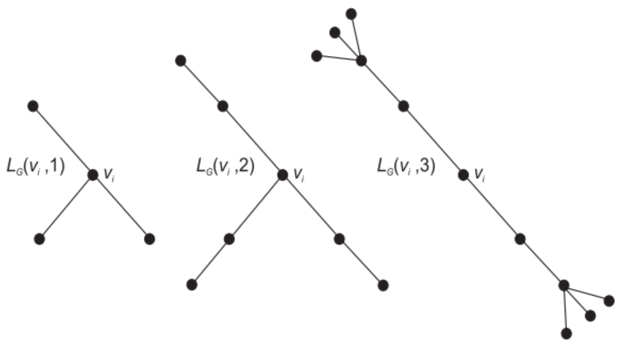

Figure 5. Local information graphs of G with respect to v_i for $j = 1,2,3$.

Definition 3.6. Assume that $l(P(L_G(v_i,j)))$ denotes the total path lengths in a local information graph $L_G(v_i,j)$. By this term we can define the information functional $f^P(v_i)$ (Dehmer and Mowshowitz 2011)

$$f^P(v_i) = \alpha^{b_1 l(P(L_G(v_i,1)))+b_2 l(P(L_G(v_i,2)))+\cdots+b_p l(P(L_G(v_i,p)))}$$

with $b_k > 0, 1 \leq k \leq p, \alpha > 0$ and b_k are arbitrary real positive numbers.

Definition 3.7. By the definition of information functional of Definition 3.6, we can give the following entropy (Dehmer and Mowshowitz 2011)

$$I_{f^P}(G) = -\sum_{i=1}^{|V|} \frac{f^P(v_i)}{\sum_{j=1}^{|V|} f^P(v_i)} \log\left(\frac{f^P(v_i)}{\sum_{j=1}^{|V|} f^P(v_i)}\right).$$

Theorem 3.8. Let G be graph. It is attained for every vertex $v_i \in V$ and information radii $1 \leq j \leq p$ (Dehmer 2008a)

$$l(P(L_G(v_i,j))) = \sum_{j=1}^{p} j.\left|S_j(v_i,G)\right|.$$

For example, we compute the $l(P(L_G(v_i,j)))$ for the local information graphs $L_G(v_i,j)$ of Figure 4.

$$l(P(L_G(v_i,1))) = 1 \times 3 = 3$$

$$l(P(L_G(v_i,2))) = 1 \times 3 + 2 \times 3 = 9$$

$$l(P(L_G(v_i,3))) = 1 \times 2 + 2 \times 2 + 3 \times 6 = 24$$

In the following definition, we introduce a new information functional and a new entropy measure which is called level entropy and is denoted by $I_\ell(T)$.

Definition 3.9. For a rooted tree T with order n, we introduce a new information functional such that

$$f(u) := L(u),$$

then by using Definition 2 we obtain the level entropy

$$I_\ell(T) = -\sum_{u\in V(T)} \frac{L(u)}{2L(T)} \log\left(\frac{L(u)}{2L(T)}\right).$$

4. Relations Between Level Based Information Functionals

In this section we give the main properties of level entropy I_ℓ and investigate the relations between the infurmation functional $L(u)$ and other level based information functional.

Lemma 4.1. Let T be a rooted tree. Assume that H is obtained from T by rooting it to a vertex $v \in V$ and n_i $(1 \leq i \leq h)$ are the number of vertices on level i. Then the level entropy functional $L(v)$ of T is computed by

$$L(v) = \sum_{i=1}^{h} i n_i.$$

Proof. H is the rooted tree obtained from the decomposition of T by rooting it to the vertex v. It is obtained that v is the root vertex of H and the number of vertices on each level is counted as showed in Figure 3. Then, the level differences between v and other vertices is computed as the position of vertices in H. Then, we have to calculate the level differences between v and n_i. Therefore

$$L(v) = n_1 + 2n_2 + \cdots + hn_h = \sum_{i=1}^{h} i n_i.$$

By the Lemma 4.1 we can write the probability distribution of any vertex $v \in V$ as

$$p_v = -\frac{L(v)}{\sum_{u \in V} L(u)} = \frac{n_1 + 2n_2 + \cdots + hn_h}{2L(T)}$$

such that n_i are the number of vertices of rooted tree H which is obtained from T by rooting it at vertex v.

Example 4.2. Let T be a rooted path $P_n: v_1 v_2 \dots v_n$. Then the level entropy functional $L(v_1)$ equals to

$$L(v_1) = \frac{1 + 2 + \cdots + (n-1)}{2\binom{n+1}{3}} = \frac{\frac{n(n-1)}{2}}{\frac{n(n+1)(n-1)}{3}} = \frac{3}{2n+2}$$

In here we note that the level entropy $I_\ell(T)$ is defined only rooted trees. It means that the decomposition of a rooted tree is a bit different from the decomposition of a cyclic graph into set of generalized trees (Mowshowitz and Dehmer 2012). If a tree is decomposed by the method as in Figure 3, there is no incident edge which is an edge between the vertices of same level or non-adjacent levels. Then, we can give the decomposition of a rooted tree T into a set of rooted trees as in the following algorithm which is similar to decomposition of a graph G to special generalized trees (Dehmer 2008b).

Algorithm 4.3. A rooted tree T is with $|V| = n$ vertices is decomposed into set of rooted trees as in the following method: Label the vertices of T from 1 to $|V|$. The set $L_S = \{1,2,\dots,|V|\}$ is called the label set. Select a desired height h and select an arbitrary label from L_S, say i. A tree is rooted at this vertex i. Now, apply the follwing steps:

1. Compute the shortest distance from vertex i to all other vertices in rooted tree T (Dijkstra Algorithm can be used for this).
2. The vertices at distance k are the k-level vertices of the rooted tree of vertex i. Select all vertices in the rooted tree at distance up to h.
3. Delete the label i from the label set L_S.
4. Repeat this procedure if L_S is not empty by choosing an arbitrary label from L_S, otherwise finish.

Lemma 4.4. The time complexity of calculation of the $I_\ell(T)$ entropy for a rooted tree T is $O(|V|^3)$.

Proof. In the computation of $I_\ell(T)$, we use the functional $L(u)$ which is depend on determining j-spheres of all vertices $u \in V$. It is required to obtain all distances $d(u, v)$ for all vertex pairs $u, v \in V$. In order to compute these distances, a shortest path algorithm can be applied (for example Dijkstra Algorithm) $|V|$ times for each vertex as a starting point. Cormen et al. (1990) obtained that Dijkstra Algorithm has time complexity $O(|V|^3)$ and this finishes the proof.

Lemma 4.5. Let T be a rooted tree. For a vertex $v \in$ V, the following relation is attained between the level entropy functional and the total path length regarding local information graph of v

$$L(v) \geq I(P(L_T(v,j)))$$

with the equality is attained if and only if every leaf of T is at distance h from v.

Proof. By Theorem 3.8, the total path length equals to following equation with height h

$$l(P(L_T(v_i,j))) = \sum_{j=1}^{h} j.\left|S_j(v_i,T)\right|.$$

If a vertex of T has no descendant on the level j, this vertex is not contained by $L_T(v,j)$. It means that the number of vertices on j-th level (n_j) in T is bigger than the number vertices at level j ($\left|S_j(v_i,T)\right|$) in the local information graph $L_T(v,j)$. Therefore

$$L(v)=\sum_{j=1}^{h} jn_j > l(P(L_T(v,j)))=\sum_{j=1}^{h} j.\left|S_j(v,T)\right|.$$

For the equality, it must be $n_j=\left|S_j(v,T)\right|$. This equality is only attained that $T=L_T(v,h)$ with height h. Consequently, each leaf of T has equal distance h from v.

5. A Comparison Between Level and Wiener Entropies

In this section we make a comparison between the level entropy and Wiener entropy for a class of rooted trees. For this purpose, we use the majorization method as in defined by Das and Shi (2017).

We consider non-increasing arrangement of each vector in R^n such that for a vector $x=(x_1,x_2,\dots,x_n)\in R^n$. Thus we have $x_1\geq x_2\geq\cdots\geq x_n$.

Definition 5.1. For $x,y\in R^n$, $x\prec y$ if

$$\begin{cases}\sum_{i=1}^{k} x_i\leq\sum_{i=1}^{k} y_i\,, i=1,2,\dots,n-1,\\ \sum_{i=1}^{n} x_i=\sum_{i=1}^{n} y_i\,.\end{cases}$$

When , $x\prec y$, x is said to be majorized by y (y majorizes x).

Let $p_\alpha(G)=\big(p_\alpha(v_1),p_\alpha(v_2),\dots,p_\alpha(v_n)\big)$ is a probably vector of the graph G with respect to a parameter α. Since the information entropy $I_\alpha(G)=-\sum_{i=1}^{n} p_\alpha(v_i)\log p_\alpha(v_i)$,it is obtained that the fuction $h(x)=-x\log x$ is a concave function for $x>0$. Then the following theorem is very useful.

Theorem 5.2. Let H and G be two non-isomorphic graphs of order n and $p_\alpha(H)$, $p_\alpha(G)$ be the probability vectors of H and G, respectively. If $p_\alpha(H)\prec p_\alpha(G)$, then we obtain that $I(G)\leq I(H)$ (Das and Shi 2017).

Lemma 5.3. Let T be a rooted tree. Then $L(T)\leq W(T)$ with the equality is attained if and only if T is a rooted path.

Proof. Since the level index equals to sum of distances bewteen the vertices of different level in a rooted tree T and Wiener index equals to sum of all distances between to any vertices of T, $L(T)\leq W(T)$ with the equality if and only if T is a rooted path.

Lemma 5.4. Let T be a rooted tree with n vertices. Then the level functional of root vertex u_0 has the relation that $L(u_0)\geq L(v)$ for any $v\in V$.

Proof. The level entropy functional of a vertex u is computed by

$$L(u) = \sum_{v \in \mathrm{V}} |l_u(T) - l_v(T)|.$$

Since the level of u_0 is 0, $L(u_0) = \sum_{v\in \mathrm{V}} l_v(T) \geq L(v)$ for any $v \in V$.

Lemma 5.5. Let T be a rooted tree and v be a most distant vertex to any vertex of T. Then the Wiener entropy functional of v has the relation that $D(v) \geq D(u)$ for any $u \in V(T)$.

Proof. The proof is clear and we omit the details.

Proposition 5.6. Let T be a rooted tree whose root vertex is one of the most distant vertices to any vertex of T, then $I_\ell(T) \leq I_w(T)$ with the equality is attained if and only if $T \cong P_n$.

Proof. We use majorization method for the proof. If T is a path, $L(T) = W(T)$ and there is one vertex at each level. Then level information functional and distance information functional of each vertex equals. Then the equality is attained if and only if T is a path.

Now assume that T is not a path. It means that T has at least two vertices which are on the same level k. By this way we can compare the information functionals $L(u_i)$ and $D(u_i)$ for each vertex $i = 0, \ldots, n-1$.

The level entropy distribution is denoted by p_ℓ and Wiener entropy distribution is deneoted by p_w of T as follows

$$p_\ell = \left(\frac{L(u_0)}{2L(T)}, \frac{L(u_1)}{2L(T)}, \ldots, \frac{L(u_k)}{2L(T)}, \ldots, \frac{L(u_{n-1})}{2L(T)}\right),$$

$$p_w = \left(\frac{D(u_0)}{2W(T)}, \frac{D(u_1)}{2W(T)}, \ldots, \frac{D(u_k)}{2W(T)}, \ldots, \frac{D(u_{n-1})}{2W(T)}\right).$$

First we take $i = 0$. It implies that u_0 is root vertex. Since the distance from root vertex to any vertex v of T equals to the level of v, then we obtain that $L(u_0) = D(u_0)$. By the definition of the information functionals $L(u_0)$and $D(u_0)$

$$\frac{L(u_0)}{\sum_{i=1}^{n-1} L(u_\mathrm{i})} = \frac{L(u_0)}{2L(T)}, \frac{D(u_0)}{\sum_{i=1}^{n-1} D(u_\mathrm{i})} = \frac{D(u_0)}{2W(T)}$$

and the inequality $L(T) < W(T)$, we obtain that

$$\frac{L(u_0)}{2L(T)} > \frac{D(u_0)}{2W(T)}$$

Now assume that $i = 1$, the vertex u_1. As mentioned for the root vertex, $L(u_1) = D(u_1)$ and we obtain that

$$\frac{L(u_1)}{2L(T)} > \frac{D(u_1)}{2W(T)}$$

The same situation goes on for the vertices which are located at first $k-1$ levels.

Now we make investigations for vertex u_k which is one of the vertices on level k. Since there is two vertices at level k, $D(u_k) = L(u_k) + 2$.

For the vertices $k < i \leq n-1,\ D(u_i) = L(u_i) + 2$.

It is understood that the first k terms (for the vertices of the level $i = 0,1, \dots, k-1$) of p_ℓ are greater than the first k terms of p_w. This situation changes with the k-th term. For a suitable term $r\ \geq k$, it must be obtained

$$\frac{D(u_r)}{2W(T)} > \frac{L(u_r)}{2L(T)}$$

because

$$\sum_{i=0}^{n-1} \frac{L(u_i)}{2L(T)} = 1, \sum_{i=0}^{n-1} \frac{D(u_i)}{2W(T)} = 1.$$

Consequently p_ℓ majorizes p_w and by Theorem 5.2, it is obtained that $I_\ell(T) \leq I_w(T)$. If the number of branch vertices will increase, the majorization will be attained more strongly and the difference between Wiener entropy and level entropy is also increased.

6. Level Based Entropy Measures of Some Trees

In this section we obtained the level based entropy measures of stars. Since caterpillar graphs is widely used in statistical analysis of graphs as mentioned in the introduction and they represent the chemically important benzenoid graphs, we obtained the level entropy of caterpillar graphs and paths.

Theorem 6.1. The level entropy of S_n is computed by the following equation

$$I_\ell(S_n) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\ \log\frac{1}{2n-2}$$

Proof. Let S_n be a rooted star with the root vertex u_0 and $n-1$ leaves. The level index of S_n equals to $n-1$, $\frac{L(u_0)}{2L(T)} = \frac{n-1}{2n-2}$ and for a leaf v $\frac{L(v)}{2L(T)} = \frac{1}{2n-2}$. By this way we compute the level entropy of S_n as follows

$$I_\ell(S_n) = -\frac{n-1}{2n-2}\log\frac{n-1}{2n-2} - (n-1) \times \frac{1}{2n-2}\log\frac{1}{2n-2}$$

$$= -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2n-2}$$

Theorem 6.2. The level decomposition entropy of S_n is computed by the following equation

$$I^V(S_n) = -\frac{2n-1}{n}\log\frac{1}{n} - \frac{n-1}{n}\log\frac{n-1}{n} - \frac{n^2-3n+2}{n}\log\frac{n-2}{n}$$

Proof. The level decomposition of S_n is attained as follows, If the decomposition is started from the root vertex, it is obtained that

$$I^V(H_1) = -\frac{1}{n}\log\frac{1}{n} - \frac{n-1}{n}\log\frac{n-1}{n}.$$

If the decomposition is started from a leaf, it is obtained that

$$I^V(H_2) = -\frac{1}{n}\log\frac{1}{n} - \frac{1}{n}\log\frac{1}{n} - \frac{n-2}{n}\log\frac{n-2}{n}$$
$$= -\frac{2}{n}\log\frac{1}{n} - \frac{n-2}{n}\log\frac{n-2}{n}$$

It is obtained the equality $I^V(H_2) = \cdots = I^V(H_n)$ for the leaves and the level decomposition entropy of S_n equals to

$$I^V(S_n) = -\frac{1}{n}\log\frac{1}{n} - \frac{n-1}{n}\log\frac{n-1}{n} - (n-1)\left[\frac{2}{n}\log\frac{1}{n} + \frac{n-2}{n}\log\frac{n-2}{n}\right]$$
$$= -\frac{2n-1}{n}\log\frac{1}{n} - \frac{n-1}{n}\log\frac{n-1}{n} - \frac{n^2-3n+2}{n}\log\frac{n-2}{n}$$

Theorem 6.3. The level entropy of rooted caterpillar tree T which is obtained from a $P_h: u_0u_1 \dots u_{h-1}$ by attaching X leaves to each vertex P_h equals to

$$-\sum_{i=1}^{h}\frac{i+(h-i)X+\left[\binom{i}{2}+\binom{h-i}{2}\right](X+1)}{2\left(hX+\binom{h+1}{3}(X+1)^2\right)}\log\frac{i+(h-i)X+\left[\binom{i}{2}+\binom{h-i}{2}\right](X+1)}{2\left(hX+\binom{h+1}{3}(X+1)^2\right)}$$

Proof. It implies that every vertex of P_h takes X leaves. By this rule, we obtain that there is u_0 on level 0, $X+1$ leaves on level i for $1 \le i \le h-1$ and X leaves on level h.

First we compute the level differences between the root u_0 and other vertices. We obtain that

$$(X+1) + 2(X+1) + \cdots + (h-1)(X+1) + hX.$$

Now, we compute the level differences between a vertex on level 1 and the vertices of following levels. Since there exists $X+1$ vertices on level 1, then it is obtained

$$(X+1)[(X+1)+2(X+1)+\cdots+(h-2)(X+1)+(h-1)X].$$

Similarly, we compute the level differences between a vertex on level 2 and the vertices of following levels. Since there exists $X+1$ vertices on level 2, then we obtain that

$$(X+1)[(X+1)+2(X+1)+\cdots+(h-3)(X+1)+(h-2)X].$$

This goes on to the vertices on level $h-1$ and the level differences between the vertices on levels $h-1$ and h equal to

$$(X+1)X.$$

Finally the level index of T equals to sum of all level differences of two any vertices by Definition 2.10 such that

$$L(T)=hX+(X+1)^2\sum_{i=1}^{h-1} i(h-i)$$

$$=hX+(X+1)^2\left[\frac{h^2(h-1)}{2}-\frac{(h-1)h(2h-1)}{6}\right]$$

$$=hX+\binom{h+1}{3}(X+1)^2$$

Now we obtain the level entropy functional of any vertex $u\in T$ on level i. There are $i-1$ levels before the level of u as the root vertex on level 0 and $X+1$ vertices on other levels. More over there are $X+1$ vertices in the following levels outside of the last level h and there are X vertices on last level h. Then the level fonctional of u equals to

$$L(u)=i+(i-1)(X+1)+(i-2)(X+1)+\cdots+(X+1)+$$

$$(X+1)+2(X+1)+\cdots+(h-i-1)(X+1)+(h-i)X$$

$$=i+\frac{i(i-1)}{2}\times(X+1)+\frac{(h-i-1)(h-i)}{2}\times(X+1)+(h-i)X$$

$$=i+(h-i)X+\left[\binom{i}{2}+\binom{h-i}{2}\right](X+1).$$

The proof is completed.

Example 6.4. By Theorem 6.3, we can compute the level entropy of a rooted path P_n. A path graph of order n has n levels with one vertices on each level. Then level entropy of P_n equals to

$$I_\ell(P_n) = -\sum_{i=0}^{n-1} \frac{\binom{i+1}{2}+\binom{n-i}{2}}{2\binom{n+1}{3}} \log \frac{\binom{i+1}{2}+\binom{n-i}{2}}{2\binom{n+1}{3}}$$

By Proposition 5.6, the level entropy and Wiener entropy of P_n are equal as $I_\ell(P_n) = I_w(P_n)$.

7. Conclusion

In this paper, we defined a new graph entropy which is called level entropy. There are some level based entropy measures in the literature. Then, we obtain some relations between the level entropy and other level based entropy measures for rooted trees. Moreover, we show that level entropy is less than or equal to Wiener entropy of a rooted tree, if the root vertex of this tree is one of the most distant vertices to any vertex. Then, there are some open problems about level entropy. We conclude the paper with these problems.

Problem 1. The relation is holds $I_\ell(T) \leq I_w(T)$ for any rooted tree T.

Problem 2. Find the extremal trees with respect to level entropy.

References

Bhutani K, Kalpathy R, Mahmoud H (2020), Average measures in polymer graphs, International Journal of Computer Mathematics: Computer Systems Theory 6 (1): 37-53. https://doi.org/10.1080/23799927.2020.1860134

Brandes U (2001), A faster algorithm for betweenness centrality, J. Math. Sociol. 25 (2): 163–177. https://doi.org/10.1080/0022250X.2001.9990249

Cao S, Dehmer M, Shi Y(2014), Extremality of degree based graph entropies, Information
Sciences 278: 22-33. https://doi.org/10.1016/j.ins.2014.03.133

Cao S, Dehmer M, Kang Z (2017), Network entropies based on independent sets and matchings, Appl. Math. Comput. 307: 265–270. https://doi.org/10.1016/j.amc.2017.02.021

Chen Z, Dehmer M, Shi Y (2014), A note on distance-based graph entropies, Entropy 16 (10): 5416–5427. https://doi.org/10.3390/e16105416

Cambie S, Dong Y (2025), On the main distance-based entropies: the eccentricity- and Wiener-entropy, Ars Mathematica Contemporanea, In Press. https://doi.org/10.26493/1855-3974.3042.4c5

Cormen TH, Leiserson CE, Rivest RL(1990), Introduction to Algorithms. Cambridge: MIT Press.

Das KC, Shi Y (2017), Some properties on entropies of graphs, MATCH Commun. Math. Comput. Chem. 78: 259-272.

Dehmer M (2008a), Information processing in complex networks: graph entropy and information functionals, Applied Mathematics and Computation 201: 82-94. https://doi.org/10.1016/j.amc.2007.12.010

Dehmer M (2008b), Information-Theoretic Concepts for The Analysis of Complex Networks, Applied Artificial Intelligence 22 (7-8): 684-706. https://doi.org/10.1080/08839510802164101

Dehmer M, Mowshowitz A(2011), A history ofgraph entropy measures, Information Sciences 181 : 57–78. https://doi.org/10.1016/j.ins.2010.08.041

Dehmer M *et al*. (2020), The Orbit-Polynomial: A Novel Measure of Symmetry in Networks, IEEE Access, vol. 8 : 36100-36112. doi: 10.1109/ACCESS.2020.2970059

Desmarais C, Mahmoud H (2021), Depts in hookings networks, Probability in Engineering and Informational Sciences 36 (4): 1-9. doi: 10.1017/S0269964821000164

Dijkstra E W (1959), A note on two problems in connection with graphs, Numerische Math. 1: 269-272.

Ghorbani M, Dehmer M, Mowshowitz A, Tao J, Emmert-Streib F (2019**)**, The Hosoya Entropy of Graphs Revisited. Symmetry 11, 1013. https://doi.org/10.3390/sym11081013

Ilić A, Dehmer M (2015), On the distance based graph entropies, Applied Mathematics and Computation 269: 647-650.

Mowshowitz A (1968), Entropy and the complexity of the graphs I: An index of the relative complexity of a graph, Bulletin of Mathematical Biophysics 30: 175–204 https://doi.org/10.1007/BF02476948

Mowshowitz A, Dehmer M (2012), Entropy and the complexity of graphs revisited, Entropy 14: 559–570. https://doi.org/10.3390/e14030559

Rashevsky N (1955), Life information theory and topology, Bulletin of Mathematical Biophysics 17: 229–235. https://doi.org/10.1007/BF02477860

Şahin B (2022), New Network Entropy: The Domination Entropy of Graphs, Information Processing Letters 174, 106195. https://doi.org/10.1016/j.ipl.2021.106195

Emmert-Streib F, Dehmer M (2007), Information theoretic measures of UHG graphs with low computational complexity. Applied Mathematics and Computation 190 (2):1783–1794.

Trucco E (1956), A note on the information content of graphs, Bulletin of Mathematical Biology 18 (2): 129–135. https://doi.org/10.1007/BF02477836

Wan P, Chen X, Tu J, Dehmer M, Zhang S, Emmert-Streib F (2020), On graph entropy measures based on the number of independent sets and matchings, Information Sciences 516: 491-504. https://doi.org/10.1016/j.ins.2019.11.020

Wasserman S, Faust K, Social network analysis: methods and applications, Structural Analysis in the Social Sciences, Cambridge University Press, 1994.

Wiener H (1947), Structural determination of paraffin boiling points, Journal of The American Chemical Society 69 (17): 17-20.